Sierpinski Gasket in a Reaction-Diffusion System

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We shall show by computer simulations that a Bonhoeffer van der Pol type reaction-diffusion system in one dimension reveals a curious spatiotemporal pattern in the motion of interacting pulses. For suitably chosen nonlinearity and parameters, the trajectory of pulses exhibits a self-similar regular pattern like a Sierpinski gasket in the space-time coordinate. This is caused by self-replication of a pulse and annihilation and/or preservation of propagating pulses upon collision. The formation of the Sierpinski gasket can be understood by mapping the time evolution of pulses to an equivalent cellular automaton. [S0031-9007(98)06956-7]

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Pulse dynamics far from equilibrium has attracted much interest recently. Computer simulations of various reaction-diffusion systems have revealed an unexpectedly rich variety of dynamical behaviors of pulses.

One of the most remarkable properties is that propagating pulses do not necessarily annihilate upon collision [1–5]. Two counterpropagating pulses interact but finally leave unchanged like solitons in an integrable system. This phenomenon, which we call preservation of pulses, occurs in some restricted parameter regime. In most of the parameter space, two pulses simply undergo pair annihilation upon collision as usual in a dissipative system. Another interesting property of pulses is self-replication which has been discovered by both computer simulations [6–8] and real experiments [9,10].

In the present Letter, we shall show that these three basic characters of pulses, i.e., pair annihilation, preservation, and self-replication, can coexist in some parameter regime and that the interplay of these components causes an interesting spatiotemporal behavior. That is, the trajectory of interacting pulses in a reaction-diffusion system produces a regular self-similar pattern like a Sierpinski gasket in the space-time coordinate. A preliminary result has been published [11]. The main purpose here is to clarify the mechanism of formation of the Sierpinski gasket by mapping the pulse dynamics to a cellular automaton.

Our model equation for the spatiotemporal evolution of interacting pulses is given by the following reaction-diffusion system:

$$\tau \frac{\partial u}{\partial t} = D_u \nabla^2 u + f(u) - v, \tag{1}$$

$$\frac{\partial v}{\partial t} = D_v \nabla^2 v + u, \tag{2}$$

where τ is the ratio of the relaxation rates of the variables u and v. The positive constants D_u and D_v are the diffusion rates of u and v, respectively. The function $f(u)$ takes the form

$$f(u) = \frac{1}{2}\left[\tanh\left(\frac{u-a}{\delta}\right) + \tanh\left(\frac{a}{\delta}\right)\right] - u, \tag{3}$$

where $0 < a < 1$ and δ are positive constants. Note that the function f becomes a piecewise linear form with $f(0) = f(a) = f(1) = 0$ in the limit $\delta \to 0$.

In our previous papers [12,13], we have studied the set of Eqs. (1) and (2) for $\tau = 1$ by computer simulations in one and two dimensions. What we have found is as follows. First of all, when the diffusion terms are absent and δ is sufficiently small, this set of equations has a subcritical Hopf bifurcation by decreasing the parameter a. Therefore the stable stationary state $u = v = 0$ and a stable limit cycle solution around it (and an unstable limit cycle in between) coexist. Second, despite the oscillatory property, the system has a stable propagating pulse solution when diffusion is present. Third, when the parameters D_v and a are sufficiently small while D_u is of the order of unity, a collision of two counterpropagating pulses causes a localized oscillatory domain which emits persistently outgoing waves.

Here we explore the pulse dynamics in the case $\tau < 1$. Throughout this paper, we set $D_u = 1$ and $\delta = 0.05$ unless stated otherwise. The Neumann boundary condition is imposed at the system boundaries.

Figure 1 summarizes the phase diagram obtained by one-dimensional simulations of (1) and (2) with (3) for $a = 0.1$. When the diffusion constant D_v is large, the inhibitor v diffuses rapidly so that a propagating pulse becomes unstable. The full line in Fig. 1 is a line above which a stable steadily propagating pulse does not exist. When the parameter τ is smaller than the broken line approximately given by $\tau \approx 0.3$, two pulses annihilate upon collision as in the ordinary case. However, preservation of pulses occurs in the region right of the broken line. A representative example is shown in Fig. 2 for $D_v = 8.5$ and $\tau = 0.31$. A pair of propagating pulses annihilates temporally upon collision but survives again and propagates in the opposite direction. It is expected that this preservation is related to that found in the sine-Gordon equation with a dissipative term [14]. In the region right of the dotted line in Fig. 1, a stable oscillatory domain is nucleated as in the case $\tau = 1$.

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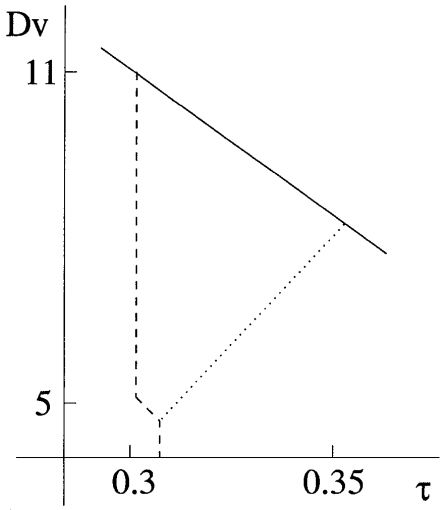

FIG. 1. The phase diagram in the D_v-τ plane. The details of the lines are given in the text.

Figure 1 was obtained numerically by incrementing the parameters by $\Delta\tau = 0.01$ and $\Delta D_v = 0.5$. Within this uncertainty, all the phase boundaries are found to be given by straight lines. We have found numerically that no stable uniform oscillation exists when $\tau < 0.3$ for $a = 0.1$. This is almost coincident with the broken line in Fig. 1. On the other hand, we have verified that the full line agrees with the stability limit of a propagating pulse, which was obtained analytically for the piecewise linear limit of (3) ($\delta \rightarrow 0$) [15]. At present, however, we do not have any definite theoretical interpretation why the dotted line is straight.

The phase diagram weakly depends on the scheme of numerical computation. However, the global property is qualitatively unaltered by changing the time increment and the space mesh size.

As mentioned above, a propagating pulse is not stable above the full line. However, a pulse can propagate transiently for some finite interval. When $\tau < 0.33$ and $D_v = 10$ such a pulse becomes small and disappears after traveling a finite distance. When $\tau > 0.33$ for $D_v = 10$, on the other hand, a pulse ceases to move after a finite time interval changing its shape and breaks up into two pulses which propagate in opposite directions. A newly generated pulse decreases its velocity and repeats the above process. An example of pair production is shown in Fig. 3 where $a = 0.1$ and $\tau = 0.34$.

A spectacular phenomenon appears in the collision of pulses generated by the self-replication. Figure 4 displays an example where a spatiotemporal evolution of pulses is plotted for $a = 0.1$ and $\tau = 0.34$. The initial conditions are $u = \exp(-x^2)$ and $v = 0$. One can see two different behaviors upon collision. When two pulses have almost the same velocity, those pulses undergo

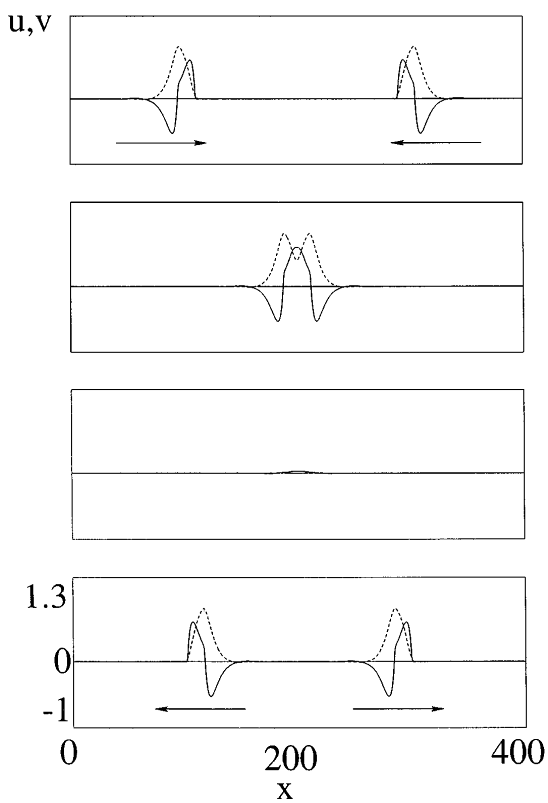

FIG. 2. Preservation of pulses upon a head-on collision for $\tau = 0.31$. The time steps are $t = 0, 12, 17, 30$ from top to bottom. The full (dotted) line indicates the profile of u (v).

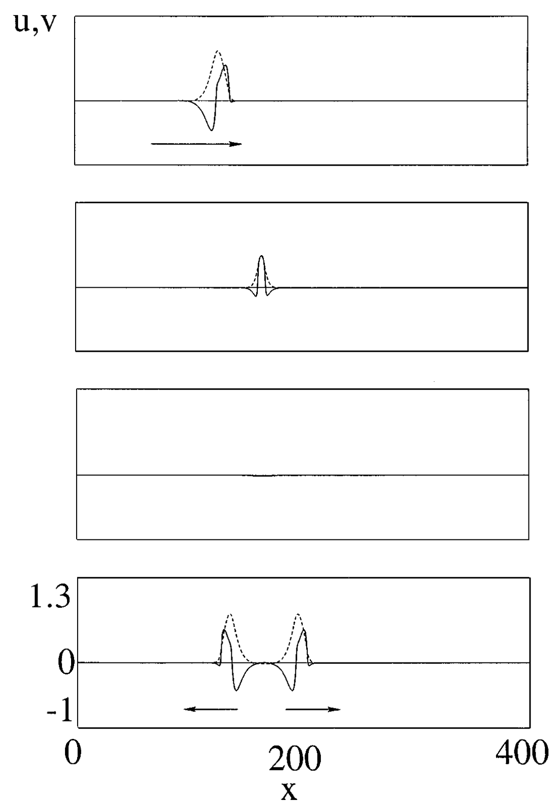

FIG. 3. Self-replication of a pulse for $\tau = 0.34$. The time steps are $t = 0, 11, 18, 25$ from top to bottom. The full (dotted) line indicates the profile of u (v).

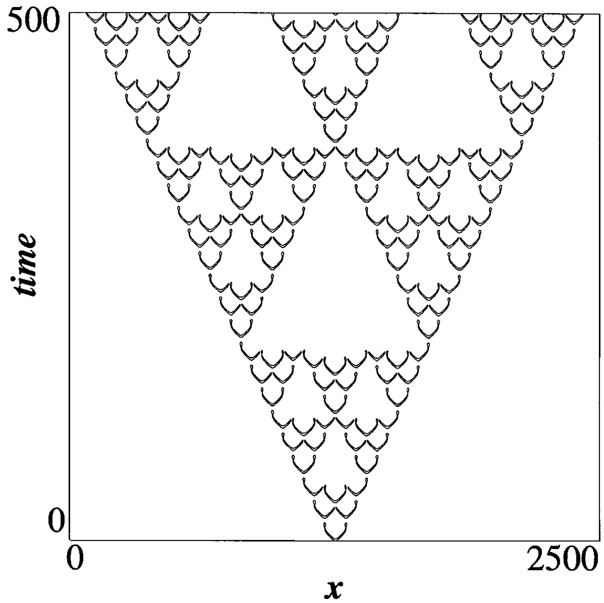

FIG. 4. Spatiotemporal pattern of interacting pulses for $\tau = 0.34$ and $D_v = 10$. The lines indicate the contour line of $u = 0.2$.

preservation as in Fig. 2. On the other hand, when the velocity difference exceeds a certain magnitude, they undergo a pair annihilation. Two pulses born at different times have a different velocity and a different shape. This asymmetry is responsible for the pair annihilation. Successive events of self-replication and preservation or annihilation of pulses cause a Sierpinski gasket pattern in the space-time coordinate. To our knowledge, a regular self-similar pattern has not been obtained so far in any partial differential equations [16].

It is well known that a Sierpinski gasket is generated by a cellular automaton [17] where both space and time are discretized. The rule can be expressed by the recurrence formula

$$a^{t+1}(i) = a^t(i - 1) + a^t(i + 1) \bmod k\,, \tag{4}$$

where i is the cell number in one dimension and $a^t(i)$ takes non-negative integers. When $k = 3$, the pattern for $a^t(i) = 1$ and 2 in the $t - i$ space exhibits the Sierpinski gasket.

We shall show that the pattern evolution in Fig. 4 is equivalent to the rule (4). Let us summarize the four basic properties of the pulse dynamics in Fig. 4. (i) A pulse stops moving after propagating for a time interval T at a distance X and then self-replicates. The velocity is a decreasing function of time. (ii) Two pulses with the same velocity exhibit preservation. That is, they annihilate when they are a distance ℓ apart, and then a pair of outgoing pulses is generated. The distance ℓ is of the order of the pulse width. There is no appreciable interaction between pulses when the distance is larger than ℓ. (iii) When the velocity is different, two pulses undergo a pair annihilation when they are a distance ℓ apart. (iv) The time dependence of the velocity of a pulse produced by a self-replication is the same as that of the one produced by a preservation.

These are schematically shown in Fig. 5 where space and time are discretized by the units X and T, respectively. For simplicity we assume that the duration of self-replication is the same as that of preservation. We start with a pair of pulses of first generation born at $t = 0$. At $t = T$, these undergo self-replication and produce four pulses of second generation. The middle two pulses undergo, by symmetry, preservation when they are a distance ℓ apart [property (ii)]. Since ℓ is finite, preservation occurs earlier, with a finite time difference ϵ, i.e., at $t = 2T - \epsilon$, than the self-replication of the other two pulses at $t = 2T$ as is shown in Fig. 5. The distance ℓ is found to be insensitive to the parameter so that the above behavior is independent of the parameters. The two pulses at the third generation produced by the preservation collide [18] and annihilate with the pulses produced by self-replication because of the asymmetry of the velocity. Therefore all the pulses except for the one at the edges of the group undergo extinction at the third generation. There are 12 pulses at the sixth generation, and, by symmetry, the pair of pulses at the middle point has the same velocity so that preservation of pulses occurs there. At the ninth generation, however, the number of pulses is 18 and hence the extinction appears again. Repeating the above processes, the Sierpinski gasket is formed as a spatiotemporal pattern. Note that any pair of pulses which undergo a simple annihilation have the age difference of ϵ.

The explanation mentioned above can be more quantified. We assign the variable $b^g(i)$ at each event which is caused by a pulse of gth generation at the discretized space point i. By an event we mean preservation, self-replication, and pair annihilation. If a pair annihilation occurs, we set $b^g(i) = 0$. As shown in Fig. 5, an event does not necessarily occur at the lattice point of the rectangular cell because of the time difference ϵ. We put $b^g(i) = n + 1$ with n a non-negative integer for events at $t = gT - n\epsilon$. An example is shown in Fig. 5 where

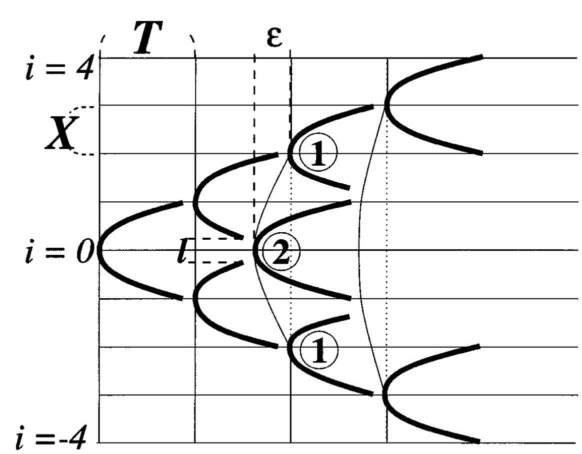

FIG. 5. Schematic trajectory of pulses. The number in a circle indicates the state variable $b^g(i)$.

self-replication of the pulses of the second generation has $b^2(2) = b^2(-2) = 1$, whereas preservation at the middle point has $b^2(0) = 2$.

The above assignment of the state variable $b^g(i)$ is equivalent with the following rule for the time evolution. The state of the next generation $b^{g+1}(i)$ is determined uniquely from $b^g(i-1)$, $b^g(i)$, and $b^g(i+1)$. A self-replication can be represented by one of the three processes: (a) $b^g(i-1) = 0$, $b^g(i) = 0$, $b^g(i+1) = m \rightarrow b^{g+1}(i) = m$; (b) $b^g(i-1) = 0$, $b^g(i) = m$, $b^g(i+1) = 0 \rightarrow b^{g+1}(i) = 0$; and (c) $b^g(i-1) = m$, $b^g(i) = 0$, $b^g(i+1) = 0 \rightarrow b^{g+1}(i) = m$ with m positive integers. In the case of collision, there are two possibilities due to the properties (ii) and (iii). That is, a symmetric collision produces a pair of pulses which is represented as (d) $b^g(i-1) = m$, $b^g(i) = 0$, $b^g(i+1) = m \rightarrow b^{g+1}(i) = m + 1$, whereas an asymmetric collision with different velocities causes annihilation of pulses, i.e., (e) $b^g(i-1) = n$, $b^g(i) = 0$, $b^g(i+1) = m \rightarrow b^{g+1}(i) = 0$ with $n \neq m$. Since we have started with a single pulse, a configuration such as $b^g(i-1) = n$, $b^g(i) = m$, $b^g(i+1) = 0$ with positive integers n and m does not exist. Note that when two pulses do not cause preservation upon collision, the difference of their ages is always equal to ϵ. Therefore one may put the restriction $n = m \pm 1$ in the above rule (e). In this way, it is proved that although the state variable $b^g(i)$ takes on any positive integer value (apart from the zero state for annihilation), the configuration of odd and even integers is identical to that of $a^t(i) = 1$ and $a^t(i) = 2$ generated by (4) with $k = 3$.

Since the emergence of the Sierpinski gasket results as a delicate balance of the properties (i)–(iv), it is indispensable to choose a suitable set of parameters. For instance, the Sierpinski gasket can be observed, in the present numerical scheme, only in the narrow region $0.338 < \tau < 0.342$ for $D_v = 10$ and $9.7 < D_v < 10.1$ for $\tau = 0.34$ starting with the same initial condition as in Fig. 4. However, the limitation of the parameter domain does not necessarily imply that the present result is an exceptional one specific to the particular reaction-diffusion system, Eqs. (1) and (2). Our expectation is that a self-similar evolution is possible when the three basic characters of pulses coexist although further investigation is necessary for this. In this respect, we mention an elasticlike collision which is one aspect of preservation of pulses. This was found first for very restricted parameters in simulations of special model equations [1–5], but later it was shown theoretically that it originates from the existence of a supercritical translational bifurcation where a motionless pulse loses stability and begins to propagate [19–21]. In this way, it was uncovered that the elasticlike collision is indeed a quite general property free from any specific model systems. Furthermore, it has also been shown that a translational bifurcation is responsible for self-replication of a domain in two dimensions [22].

In summary, we have shown that a regular self-similar pattern can be self-organized in a reaction-diffusion system. The relation with a cellular automaton is elucidated. It is found that the interplay of preservation, pair annihilation, and self-replication of pulses plays the decisive role of formation of a Sierpinski gasket pattern. Finally, we emphasize that formation of the Sierpinski gasket in a reaction-diffusion system described by a partial differential equation provides us with a possibility that such a regular pattern generated so far only by an artificial model system of cellular automata can be observed in a real experiment.

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[1] H. Sevcikova and M. Marek, in *Bifurcation and Chaos*, edited by R. Seydel *et al.* (Birkhauser Verlag, Basel, 1991).
[2] V. Petrov, S. K. Scott, and K. Showalter, Philos. Trans. R. Soc. London A **347**, 631 (1994).
[3] K. Krisher and A. Mikhailov, Phys. Rev. Lett. **73**, 3165 (1994).
[4] M. Mimura and M. Nagayama, SIAM J. Appl. Math. (to be published).
[5] S. Kawaguchi and M. Mimura, SIAM J. Appl. Math. (to be published).
[6] J. E. Pearson, Science **261**, 189 (1993).
[7] W. N. Reynolds, J. E. Pearson, and S. Ponce-Dawson, Phys. Rev. Lett. **72**, 2797 (1994).
[8] W. N. Reynolds, S. Ponce-Dawson, and J. E. Pearson, Phys. Rev. E **56**, 185 (1997).
[9] K. J. Lee, W. D. McCormick, J. E. Pearson, and H. L. Swinney, Nature (London) **369**, 215 (1994).
[10] K. J. Lee and H. L. Swinney, Phys. Rev. E **51**, 1899 (1995).
[11] Y. Hayase, J. Phys. Soc. Jpn. **66**, 2584 (1997).
[12] R. Kobayashi, T. Ohta, and Y. Hayase, Phys. Rev. E **50**, R3291 (1994).
[13] T. Ohta, Y. Hayase, and R. Kobayashi, Phys. Rev. E **54**, 6074 (1996).
[14] M. Argentina, P. Coullet, and L. Mahadevan, Phys. Rev. Lett. **79**, 2803 (1997).
[15] A. Ito and T. Ohta, Phys. Rev. A **45**, 8374 (1992).
[16] After submitting this paper for publication, Professor H. Chate kindly informed us of the fact that an almost regular self-similar pattern has been obtained in a complex Ginzburg-Landau equation with and without external fields. See, for instance, H. Chate, A. Pikovsky, and O. Rudzick, Physica D (Amsterdam) (to be published).
[17] S. Wolfram, Rev. Mod. Phys. **55**, 601 (1983).
[18] It is shown from the facts that ℓ is finite and the velocity is a decreasing function that this collision of pulses occurs before their own self-replication.
[19] T. Ohta and J. Kiyose, J. Phys. Soc. Jpn. **65**, 1967 (1996).
[20] T. Ohta, J. Kiyose, and M. Mimura, J. Phys. Soc. Jpn. **66**, 1155 (1997).
[21] M. Nagayama, M. Mimura, and T. Ohta (unpublished).
[22] A. Hagberg and E. Meron, Chaos **4**, 477 (1994).