Neutron-³H and Proton-³He Zero Energy Scattering

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The Kohn variational principle and the (correlated) hyperspherical harmonics technique are applied to study the n-³H and p-³He scattering at zero energy. Predictions for the singlet and triplet scattering lengths are obtained for nonrelativistic nuclear Hamiltonians including two- and three-body potentials. The calculated n-³H total cross section agrees well with the measured value, while some small discrepancy is found for the coherent scattering length. For the p-³He channel, the calculated scattering lengths are in reasonable agreement with the values extrapolated from the measurements made above 1 MeV. [S0031-9007(98)06915-4]

In the last few years the scattering of nucleons by deuterons has been the subject of a large number of investigations. This scattering problem is in fact a very useful tool for testing the accuracy of our present knowledge of the nucleon-nucleon (NN) and three nucleon ($3N$) interactions. Noticeable progress has been achieved, but a number of relevant disagreements between theoretical predictions and experimental results remains to be solved [1,2].

It is therefore of interest to extend the above mentioned analysis to four nucleon scattering processes. In this case, an important goal for both theoretical and experimental analyses is to reach a precision comparable to that achieved in the N-d case. This is particularly challenging from the theoretical point of view, since the study of $A = 4$ systems is noticeably more complicated than the $A = 3$ one. Recently, accurate calculations of the alpha particle binding energy B_4 have been achieved [3–5]. It has been shown that, with $NN + 3N$ potential models fitting the ³H binding energy, no four-nucleon potential seems necessary to reproduce the experimental value of B_4 [3]. Therefore, it is expected that NN and $3N$ interactions should be sufficient to describe the four nucleon scattering processes too. Thus, discrepancies between theory and experiment would be useful to gain further information on the nuclear interaction. For example, the polarization observables in the reaction p-³H are believed to be very sensitive to the spin-orbit interactions [6]. Moreover, four nucleon reactions play an important role also in astrophysics and other subfields of physics.

In this Letter, the problem of n-³H and p-³He zero-energy scattering is studied. The aim is to obtain accurate estimates of the corresponding scattering observables by using NN and $3N$ realistic interactions. The relevant quantities in n-³H zero energy scattering are the singlet a_s and triplet a_t scattering lengths. They can be obtained from the experimental values of the total cross section σ_T and the coherent scattering length a_c,

$$\sigma_T = \pi(|a_s|^2 + 3|a_t|^2), \qquad a_c = \tfrac{1}{4} a_s + \tfrac{3}{4} a_t. \quad (1)$$

The n-³H cross section has been accurately measured over a wide energy range and the extrapolation to zero energy does not present any problems. The value obtained is $\sigma_T = 1.70 \pm 0.03$ b [7]. The coherent scattering length has been measured by neutron-interferometry techniques. The most recent values reported in the literature have been obtained by the same group; they are $a_c = 3.82 \pm 0.07$ fm [8] and $a_c = 3.59 \pm 0.02$ fm [9], the latter value being obtained with a more advanced experimental arrangement. Recently, the estimation of $a_c = 3.607 \pm 0.017$ fm has been obtained from p-³He data by using an approximate Coulomb-corrected R-matrix theory [10].

The corresponding quantities for p-³He scattering are more difficult to evaluate. Approximate values have been determined from effective range extrapolations to zero energy of data taken above 1 MeV, and therefore suffer large uncertainties [11,12].

From the theoretical point of view, the problem of the scattering of four nucleons has been considered for a long time (see Ref. [13], and references therein). The most widely used techniques are based on the Faddeev-Yakubovsky (FY) equations [14–16] and the Kohn-Hulthén variational principles [17]. In the latter case, the resonating group method has been used to parametrize the wave function (WF) [18,19], but also the expansion of the WF on a hyperspherical harmonic (HH) basis has been investigated [20]. Calculations using the FY and HH techniques, which allow for the full description of the four-body dynamics, were performed by using simple central or separable potentials. Only recently, the FY equations have been solved by adopting realistic NN potentials [21].

In the present paper, the wave functions of the scattering states are expanded in terms of the correlated hyperspherical harmonic (CHH) basis [22] and the Kohn-Hulthén variational principles are applied. Such a technique has been successfully used in the study of the N-d scattering below and above the deuteron breakup threshold. The present calculations follow exactly the same line followed in the N-d case described in Ref. [23]. Let us consider the

p-^{3}He scattering; the case of n-^{3}H scattering can be easily obtained in the limit $e^2 \to 0$, where e is the unit charge (see also Ref. [24]). The WF with total angular momentum J, parity Π, and total isospin T, T_z can be written as

$$\Psi_{LS}^{\gamma} = \Psi_C^{\gamma} + \Phi_{LS}^{\gamma}, \qquad (2)$$

where the index γ denotes hereafter the set of quantum numbers J, Π, T, T_z. The first term Ψ_C^{γ} of Eq. (2) must be sufficiently flexible to guarantee a detailed description of the "core" of the system, when all the particles are close to each other and the mutual interaction is large; Ψ_C^{γ} goes to zero when the p-^{3}He distance r_p increases. It has been expanded in terms of CHH basis functions, following the procedure discussed in detail in Ref. [5].

The second term Φ_{LS}^{γ} describes the asymptotic configuration of the system, for large r_p values, where the nuclear p-^{3}He interaction is negligible. The quantum number L is the relative orbital angular momentum; S is the spin obtained by coupling the spin $1/2$ of ^{3}He to the spin of the fourth nucleon. The angular momenta L and S are coupled to give the total angular momentum J. In the present study the total isospin is $T = 1$. The function Φ_{LS}^{γ} must be the solution of the two-particle Schrödinger equation appropriate for large r_p values. It is convenient to introduce the following surface functions

$$\Omega_{LS\gamma}^{(\lambda)} = \sum_{i=1}^{4} \{Y_L(\hat{r}_i)[\Phi_{jk\ell}\chi_i]_S\}_{JJ_z}[\xi_{jk\ell}\xi_i]_{TT_z}\mathcal{R}_L^{(\lambda)}(r_i), \qquad (3)$$

where the product $\Phi_{jk\ell} \times \xi_{jk\ell}$ is the WF $\Psi_{^3\mathrm{He}}$ of the ^{3}He bound state (in the case of n-^{3}H scattering, it is the WF $\Psi_{^3\mathrm{H}}$ of the ^{3}H bound state). They are normalized to unity and are antisymmetrical for the exchange of any pair of particles j, k, and ℓ. Both $\Psi_{^3\mathrm{He}}$ and $\Psi_{^3\mathrm{H}}$ have been determined as discussed in Ref. [23] by using the CHH expansion for a three-body system. Within this scheme the WF and the binding energy B_3 are determined with high accuracy. For example, the B_3 evaluated for the different potential models considered in this paper agree within a few keV to the corresponding results obtained by solving the Faddeev equations [25,26].

In Eq. (3), the spin (isospin) function of the unbound nucleon i is denoted by χ_i (ξ_i). Moreover, r_i is the distance between nucleon i and the center of mass of ^{3}He. The functions $\mathcal{R}_L^{(\lambda)}(r_i)$ of Eq. (3) can be taken to be the regular ($\lambda \equiv R$) and irregular ($\lambda \equiv I$) radial solutions of the two-body Schrödinger equation without nuclear interaction. They are analogous to those used in N-d scattering [23].

With the above definitions, the asymptotic WF is written as

$$\Phi_{LS}^{\gamma} = \Omega_{LS\gamma}^{(R)} + \sum_{L'S'} {}^{\gamma}\widetilde{R}_{LL'}^{SS'}\Omega_{L'S'\gamma}^{(I)}, \qquad (4)$$

where the matrix element ${}^{\gamma}\widetilde{R}_{LL'}^{SS'}$ gives the relative weight between the regular and the irregular $L'S'$ components.

These elements are the reactance matrix (R matrix) elements, except for some numerical factors [23]. The eigenvalues of the R matrix are $\tan\delta_{LS}$, where δ_{LS} are the eigenphase shifts of the $^{2S+1}L_{J\Pi TT_z}$ wave.

The convergence of the expansion of the internal part Ψ_C^{γ} is conveniently studied by grouping the functions of the basis in "channels" (a given channel contains CHH states with the same angular-spin-isospin quantum numbers). It is very useful to consider first the channels with orbital angular momentum values as low as possible. One channel at a time is included in the expansion of Ψ_C^{γ}; the number of the CHH functions belonging to that channel is increased until convergence is reached. If the contribution of that particular channel is found to be sizable, the corresponding CHH functions are retained in the expansion; otherwise, they are rejected. Then, others channels are added and the convergence studied in terms of the total number of channels N_c. This procedure results to be effective since (i) the value of N_c can be kept rather low, and (ii) a small number of CHH functions is sufficient, except for few channels. In particular, for the states (S wave, $T = 1$) considered here, the number of channels included finally in the wave functions is rather small ($N_c \approx 6$–8). This is due mainly to the Pauli principle which prevents the overlap of the four nucleons. As a consequence, the internal part is rather small and does not require a large number of channels.

The quantities to be determined in the WF (2) are the hyperradial functions entering the HH expansion of the internal part Ψ_C^{γ}, and the matrix elements ${}^{\gamma}\widetilde{R}_{JJ'}^{SS'}$. For these, the Kohn or the Hulthén variational principles have been used. The Kohn variational principle establishes that the following functionals,

$$[{}^{\gamma}\widetilde{R}_{LL'}^{SS'}] = {}^{\gamma}\widetilde{R}_{LL'}^{SS'} - \frac{M}{\sqrt{6}\,\hbar^2}\langle\Psi_{L'S'}^{\gamma}|H - E|\Psi_{LS}^{\gamma}\rangle, \qquad (5)$$

where ${}^{\gamma}\widetilde{R}_{LL'}^{SS'}$ are the trial parameters entering Eq. (4), must be stationary with respect to variations of all the trial parameters of the WF. In Eq. (5), E is the total (c.m.) energy and M the nucleon mass.

The form of the equations then derived and the procedure to solve them is completely analogous to those of Ref. [23] and is not repeated here. With the Hulthén variational principle the asymptotic function is written in the form

$$\Phi_{LS}^{\gamma} = \Omega_{LS\gamma}^{(I)} + \sum_{L'S'} {}^{\gamma}\widetilde{U}_{LL'}^{SS'}\Omega_{L'S'\gamma}^{(R)}, \qquad (6)$$

where $\widetilde{U} = \widetilde{R}^{-1}$. The Kohn and Hulthén variational principles lead to essentially different equations. Therefore, if the solutions in the two cases turn out to be close to each other, we are quite confident that they are close to the true solution.

The results for the singlet and triplet scattering lengths for n-^{3}H scattering are given in Table I, as a function of the number of channels included in the WF. The

TABLE I. Singlet a_s and triplet a_t S-wave scattering lengths (fm) for n-^{3}H zero-energy scattering calculated with the AV14 potential and the Kohn (rows labeled K) or Hulthén (row labeled H) variational methods. N_c is the number of channels included in the CHH expansion of the wave functions (the case $N_c = 0$ corresponds to including in the WF only the asymptotic terms). The last row reports the results obtained in Ref. [21] by solving the FY equations.

Method	N_c	a_s	N_c	a_t
K	0	4.38	0	3.87
K	1	4.33	2	3.82
K	3	4.33	4	3.82
K	4	4.32	6	3.80
K	6	4.32	8	3.80
H	6	4.32	8	3.80
FY		4.31		3.79

potential adopted in this case is the AV14 interaction [27], so that a direct comparison with the results obtained in Ref. [21] by solving the FY equations can be made. From an inspection of the table, the rapid convergence with N_c is evident; this fact reflects that (i) the scattering lengths are mainly determined by the asymptotic part and (ii) the CHH expansion basis is very effective. Moreover, there is a strict agreement between the converged values of the scattering lengths obtained by means of the Kohn and the Hulthén variational principles. Both estimates compare very well with the FY results of Ref. [21], which is a strong signal of the good accuracy of both calculations.

The calculated singlet and triplet n-^{3}H scattering lengths corresponding to different potential models are plotted versus the corresponding ^{3}H binding energy in Fig. 1. The most recent experimental values [9,10] of a_s and a_t have also been reported. The models including only NN forces are the AV14 [27], AV8 [28], and AV18 [29] potentials. Including 3N forces we have the AV14 + Urbana model VIII (AV14UR) [30], AV18 + Urbana model IX (AV18UR) [3], AV14 + Brazil with $\Lambda = 5.6 m_\pi$ (AV14BR1) and AV14 + Brazil with $\Lambda = 5.8 m_\pi$ (AV14BR2) [31]. In the AV14UR and AV18UR models, one of the parameters of the 3N potentials was chosen so that to reproduce the experimental ^{3}H binding energy value $B_3 = 8.48$ MeV. The AV14BR1 and AV14BR2 models have been chosen so as to give slightly larger B_3 values. It should be noted that all the results for the singlet (triplet) scattering length fall essentially on a straight line. However, the experimental values extracted from the data do not lie on the theoretical curves. This disagreement is related to a rather small discrepancy between the calculated and measured coherent scattering lengths, as will be shown below.

The calculated total cross section and coherent scattering length for the AV14UR and AV18UR models are compared with the experimental values [7–10] in Table II. These two potential models are chosen since
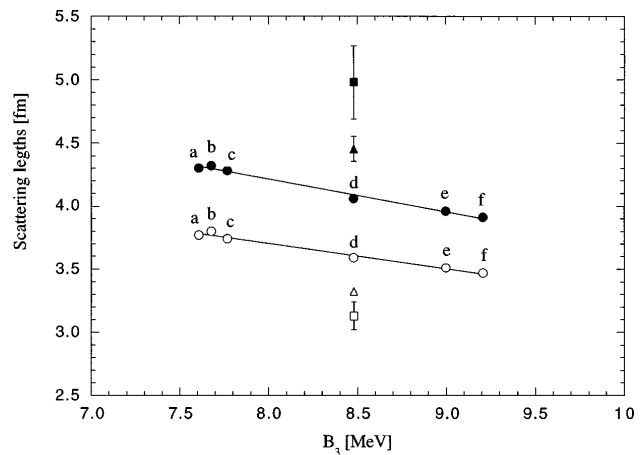

FIG. 1. Singlet (full symbols) and triplet (open symbols) scattering lengths plotted against the ^{3}H binding energy. Circles labeled by a, b, c, d, e, and f correspond to the AV18, AV14, AV8, AV18UR, AV14BR1, and AV14BR2 models, respectively. The AV14UR and AV18UR model predictions are almost coincident. The squares (triangles) are the experimental values of Ref. [9] (Ref. [10]). The straight lines are linear fits of the theoretical results.

they well reproduce the experimental B_3 value, and meaningful comparisons with the scattering data extracted from experiments can then be performed. From inspection of Table II, it can be concluded that there is a satisfactory agreement between the calculated and the measured values of σ_T. The calculated coherent scattering lengths, differ, however, by about 3% from the experimental values. This small discrepancy gives rise to the large differences in the scattering lengths, when these are determined from the relations given in Eq. (1). In fact, in the a_s, a_t plane, the ellipse corresponding to the experimental values of the total cross section $\sigma_T = 1.7$ b, and the straight line corresponding to the coherent scattering length $a_c = 3.7$ fm are almost tangent. Therefore, a slight change in the a_c value produces a large variation of a_s and a_t. This is also the reason for the large uncertainty in the values of a_s reported in Fig. 1.

The ^{3}He binding energy $B_3(^3$He) and the p-^{3}He scattering lengths as determined with the AV18 and AV18UR models are presented in Table III, together with the available experimental data [11,12]. It should

TABLE II. Total cross section σ_T (b) and coherent scattering length (fm) for n-^{3}H zero-energy scattering calculated with the AV14UR and AV18UR potential models. The last rows report the experimental values.

Model	σ_T	a_c
AV14UR	1.74	3.71
AV18UR	1.73	3.71
Expt.	1.70 ± 0.03 [7]	3.82 ± 0.07 [8]
		3.59 ± 0.02 [9]
		3.607 ± 0.017 [10]

TABLE III. ^{3}He binding energy B_3 (MeV) and singlet a_s and triplet a_t s-wave scattering lengths (fm) for p-^{3}He scattering calculated with the AV18 and AV18UR potential models. The last rows report the experimental values.

Model	B_3	a_s	a_t
AV18	6.93	12.9	10.0
AV18UR	7.74	11.5	9.13
Expt.	7.72	10.8 ± 2.6 [12]	8.1 ± 0.5 [12]
			10.2 ± 1.5 [11]

be remarked that, in contrast with the AV18UR model, the AV18 potential does not reproduce correctly the experimental value of $B_3(^3\text{He})$. More in general, it has been verified that the scattering length values show a scaling property analogous to that found in the n-^{3}H case. In Table III, the available experimental values have also been reported. However, it should be observed that (i) such experimental values have been extrapolated to zero energy from measured data taken above 1 MeV; (ii) the quoted "error bars" include only statistical and not systematical uncertainties [32]. The p-^{3}He experimental scattering lengths therefore suffer large uncertainties, even somewhat bigger than those reported in the table. By inspection of the table it can be concluded that the agreement between the AV18UR predictions and the experimental values is reasonably satisfactory and that it would be very useful to have a more accurate experimental determination of a_s and a_t. Finally, it should be noted that the p-^{3}He scattering lengths are larger than the corresponding values found in the n-^{3}H case. This result is quite similar to that found in s-wave N-d scattering in the quartet spin state.

In conclusion, accurate predictions of the n-^{3}H and p-^{3}He zero energy scattering lengths with realistic Hamiltonians including NN and $3N$ potentials have been produced. The Kohn-Hulthén variational principle and the correlated hyperspherical harmonics technique were used to solve the four-body problem and to calculate the quantities of interest. The singlet and triplet scattering lengths for n-^{3}H scattering were found to lie on straight lines when plotted against the ^{3}H binding energy for a variety of potential models. Our total cross section agrees well with the measured value, while some discrepancy is found in the comparison of the coherent scattering length values quoted in the literature. This is somewhat surprising, since the corresponding quantity in N-d scattering is well reproduced by the theory [23], and the same was expected for the four-nucleon case.

Although low-energy p-^{3}He and n-^{3}H experiments are difficult, we hope that the present work might inspire further efforts in this area.

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