Wess-Zumino Terms in Supersymmetric Gauge Theories

Aneesh V. Manohar
Department of Physics, University of California at San Diego, 9500 Gilman Drive, La Jolla, California 92093-0319
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The Wess-Zumino term is constructed for supersymmetric QCD with two colors and flavors and is shown to correctly reproduce the anomalous Ward identities. Supersymmetric QCD is also shown not to have topologically stable Skyrmion solutions because of baryon flat directions, which allow them to unwind. The generalization of these results to other supersymmetric theories with quantum modified constraints is discussed. [S0031-9007(98)06906-3]

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The configuration space of zero-energy states of a supersymmetric gauge theory is known as the moduli space $\mathcal{M}$, and is parametrized by the expectation values of gauge invariant composite fields subject to constraints. If $\mathcal{M}$ has nontrivial topology, there can exist topological terms in the effective action such as a Wess-Zumino term [1,2]. It is also possible to have topologically stable field configurations, such as Skyrmions or vortices [2,3]. The early work on Wess-Zumino terms in supersymmetric gauge theories [4–7] was done before the recent work of Seiberg and others elucidating the structure of the quantum moduli spaces [8]. The existence of topological terms is reexamined in light of these results. Supersymmetric QCD with two colors and flavors is the simplest example of a theory which has a quantum deformed moduli space with nontrivial topology. The Wess-Zumino term for this theory is studied in this paper. It is also shown that this theory does not have topologically stable Skyrmion solutions. The generalization of these results to Sp$(2n)$ theories is given at the end of this paper. Similar results should also hold for other quantum deformed theories [9].

The structure of the moduli space of supersymmetric QCD in $3 + 1$ dimensions depends on the number of colors N and flavors F. For $F < N$ the low-energy description is in terms of the expectation value of gauge invariant mesons M^i_j, and the effective theory has a nonperturbative superpotential

$$W = (N - F)\left(\frac{\Lambda^{3N-F}}{\det M}\right)^{1/(N-F)},$$

where Λ is the nonperturbative scale parameter of the theory. The quantum theory is unstable, with $\det M \to \infty$. The moduli space for $\det M \neq 0$ is isomorphic to the group GL$(F,\mathbb{C})$. This example has been studied in detail in the literature [4–7], and the analysis will not be repeated here. The moduli space $\mathcal{M} =$ GL$(F,\mathbb{C})$ has a Wess-Zumino term and supports stable Skyrmion solutions. Skyrmions in supersymmetric σ models have also been studied [10].

The cases we will examine are $F = N$ and $F = N + 1$, where the quantum moduli spaces have recently been constructed [8]. For $N = F + 1$, the moduli space [8] is given by the expectation values of gauge invariant mesons M^i_j, baryons B_i, and antibaryons $\tilde{B}^j$. Here $i, j = 1, \ldots, F$ are flavor indices. There are nontrivial constraints among the basic invariants,

$$B_i M^i_j = 0, \qquad M^i_j \tilde{B}^j = 0, \qquad \mathrm{cof}^j_i\, M = B_i \tilde{B}^j, \tag{1}$$

where $\mathrm{cof}^j_i\, M$ is the cofactor of the ij entry of M. These constraints are precisely the same as those obtained by minimizing the superpotential

$$W = \frac{B_i M^i_j \tilde{B}^j - \det M}{\Lambda^{2N-1}}. \tag{2}$$

This theory clearly has a topologically trivial moduli space. One can make a deformation retract of the moduli space to the origin $M = B = \tilde{B} = 0$ since all the constraints are homogeneous. It is therefore not possible to construct a topological term in the effective action. Obviously, a similar result holds for any theory whose moduli space is given by gauge invariants subject to homogeneous constraints, such as s-confining theories [11], or any theory whose moduli space has no constraints, such as those with an affine moduli space [12]. In these theories, the flavor anomalies of the gauge-invariant composites agree with those of the microscopic fields, so a Wess-Zumino term is not required in the low-energy theory.

The interesting case is supersymmetric QCD with $F = N$; the $F = N = 2$ case will be studied here. Since the $\mathbf{2}$ and $\bar{\mathbf{2}}$ representations of SU(2) are equivalent, the quarks and antiquarks can be combined to form four SU(2) doublets. The flavor symmetry of the theory is SU(4) $\times$ U$(1)_R$. The baryon number is part of the SU(4) symmetry. The mesons and baryons can be combined into a single 4×4 antisymmetric matrix V,

$$V = \left(\begin{array}{cc|c} 0 & B & \\ -B & 0 & M \\ \hline & & 0 \quad \tilde{B} \\ -M^T & & -\tilde{B} \quad 0 \end{array}\right), \tag{3}$$

which transforms as the two-index antisymmetric tensor under flavor SU(4) and has zero R charge. The quantum constraint is [8]

$$\mathrm{Pf} V = \Lambda^4, \tag{4}$$

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where Pf is the Pfaffian. The constraint can be written as

$$B\tilde{B} - \det M = B\tilde{B} - M_{11}M_{22} + M_{12}M_{21} = \Lambda^4. \quad (5)$$

It is straightforward to determine the topology of the quantum moduli space $\mathcal{M}$ given by V subject to the constraint Eq. (4). The SU(4) group is equivalent to SO(6), and V is the **6** dimensional (i.e., vector) representation of SO(6), which can be denoted by $(X_1, \ldots, X_6)$, where the X_i are linear combinations of the V_{ij}. The constraint Eq. (4) is the SO(6) invariant constraint

$$\sum_{i=1}^{6} X_i^2 = \Lambda^4, \quad (6)$$

and the moduli space $\mathcal{M}$ is the surface in $\mathbb{C}^6$ given by Eq. (6). It is straightforward to show that there is a deformation retract of $\mathcal{M}$ onto the real section given by taking X_i real, i.e., the five-sphere S^5.

The homotopy and cohomology groups of $\mathcal{M}$ are identical to those of S^5. In particular, $H^5(\mathcal{M}) = \mathbb{Z}$ and $\pi_3(\mathcal{M}) = 0$, so that one can write down a Wess-Zumino term, but there are no topologically stable Skyrmion solutions. By analogy with QCD (which has stable Skyrmion solutions), one can write down a "Skyrmion" field configuration $V(\mathbf{x})$ which is a static field configuration from $S^3 \to \mathcal{M}$,

$$B = \tilde{B} = 0, \qquad M = \exp[i\tau \cdot \hat{\mathbf{x}} F(|x|)], \quad (7)$$

with $F(\infty) = 0$, $F(0) = \pi$. The Skyrmion has a nontrivial winding number if one looks only at the subspace $B = \tilde{B} = 0$, $\det M \neq 0$, but can unwind because of the baryon directions B and $\tilde{B}$. It is easy to explicitly write down a sequence of field configurations that go through the point $B\tilde{B} = 1$, $\det M = 0$, and allows the Skyrmion to unwind. This result generalizes to $F = N > 2$, where the moduli space is given by a $N \times N$ matrix M, and baryons B and $\tilde{B}$ with the quantum constraint [8]

$$\det M - \tilde{B}B = \Lambda^{2N}. \quad (8)$$

[The difference in sign from Eq. (5) has to do with the relation between the **2** and $\overline{\mathbf{2}}$ representations of SU(2) and is unimportant.] A Skyrmion Eq. (7) embedded in the first 2×2 block of M can unwind because of the baryon directions.

A Wess-Zumino functional Γ on $\mathcal{M}$ can be defined following the method used by Witten [2] for QCD. A field configuration $V(x)$ is a map $V: S^4 \to \mathcal{M}$ from spacetime to the moduli space. Since $\pi_4(\mathcal{M}) = 0$, the four-surface in $\mathcal{M}$ given by the image of spacetime under V is the boundary of a five-surface Σ_5 in $\mathcal{M}$. The Wess-Zumino functional is given by integrating a closed (but not exact) five-form ω_5 defined on $\mathcal{M}$ over the five-surface $\Sigma_5 \in \mathcal{M}$. As for QCD, one finds that since $\pi_5(\mathcal{M}) = \mathbb{Z}$, the Wess-Zumino action is ambiguous. The ambiguity in Γ is an integer times the integral of ω_5 over the five-sphere that generates $\pi_5(\mathcal{M})$. The ambiguity is irrelevant for the quantum theory provided $\exp i\Gamma$ is well defined. This determines the Wess-Zumino term to be

$$\Gamma = \frac{1}{240\pi^2} \operatorname{Im} \int_{\Sigma_5} \operatorname{Tr}(V^{-1} dV)^5. \quad (9)$$

The term in the effective action is $n\Gamma$, where n is an integer. The Wess-Zumino term is well defined, since the constraint Eq. (4) implies that V is invertible. The coefficient is fixed by requiring that the integral over the five-sphere is 2π. Equation (9) gives the bosonic part of the Wess-Zumino action; one can always make the action supersymmetric by adding fermionic components and writing the Wess-Zumino term as a D term [4–7]. In the remainder of the paper, we will concentrate only on the bosonic part of the Wess-Zumino term, since that is the piece relevant for the anomalous Ward identities. The Wess-Zumino term Eq. (9) has been written using a holomorphic five-form. For a discussion of why this is possible, see section 5 of Ref. [4]. An explicit construction of the supersymmetric Wess-Zumino action given a holomorphic five-form can be found in Ref. [13].

The Wess-Zumino term has been written as the imaginary part of the integral in Eq. (9). Only the imaginary part has a quantized coefficient and contributes to the anomalous Ward identities. The real part is an allowed term in the effective action, and does not have a quantized coefficient since its integral over S^5 vanishes. The real part vanishes in QCD, because the Wess-Zumino action is written as an integral of the form Eq. (9), with $V \to U$, a unitary matrix. Here V is not unitary, and the integral has both real and imaginary parts.

The integer n multiplying Eq. (9) in the effective action is fixed by requiring that the low-energy theory reproduce all the flavor anomalies of the microscopic theory. (The microscopic theory refers to the theory written in terms of quarks and gluons, and the low-energy theory refers to the theory written in terms of gauge invariant mesons and baryons.) The microscopic theory has a $SU(4) \times U(1)_R$ flavor symmetry. The $U(1)_R$, $U(1)_R^3$, and $SU(4)^2 U(1)_R$ anomalies match between the microscopic and low-energy theories when computed using the massless fermions in the two theories, but the $SU(4)^3$ anomalies do not match. This was the original motivation for introducing the quantum deformation Eq. (4) in the moduli space $\mathcal{M}$. The SU(4) symmetry is broken at all points on $\mathcal{M}$, so the $SU(4)^3$ anomalies computed using the massless fermions of the microscopic and low-energy theories need not match. Nevertheless, the anomalous Ward identities must be satisfied [14]. Anomalous Ward identities get contributions from massless fermions and Goldstone bosons [15], so the $SU(4)^3$ Ward identity gets an additional Goldstone boson contribution from the Wess-Zumino term, which fixes n.

The contribution of the Wess-Zumino term can be determined by turning on weakly coupled background gauge fields for the $SU(4) \times U(1)_R$ flavor symmetry, and studying the variation of the Wess-Zumino term under local flavor symmetry transformations. The variation of Γ under an infinitesimal local $SU(4) \times U(1)_R$ transformation is

$$\delta\Gamma = \frac{1}{48\pi^2}\int_{\partial\Sigma_5} \mathrm{Tr}[d\epsilon^T (V^{-1}dV)^3 - d\epsilon(dVV^{-1})^3], \\ = -\frac{1}{24\pi^2}\int_{\partial\Sigma_5} \mathrm{Tr}\, d\epsilon(dVV^{-1})^3, \tag{10}$$

where $\epsilon = \epsilon^a T^a$ is an SU(4) generator, so that the Wess-Zumino term contributes to the SU(4) flavor current,

$$j^a_\mu = \frac{1}{24\pi^2}\epsilon_{\mu\nu\alpha\beta}\mathrm{Tr}T^a(\partial^\nu VV^{-1})(\partial^\alpha VV^{-1})(\partial^\beta VV^{-1}). \tag{11}$$

The field V does not transform under $U(1)_R$, so that one might naively think that the Wess-Zumino term does not contribute to the R current. However, for QCD, Witten pointed out that the Wess-Zumino term contributes to the baryon number current even though it is written in terms of mesons which have zero baryon number. This subtlety does not occur for the $U(1)_R$ current. The possible Wess-Zumino contribution to the R current is given by using Eq. (11), and replacing $\epsilon^a T^a \to \epsilon \mathbb{1}$, as for QCD. The resulting current vanishes, since

$$\mathrm{Tr}(dVV^{-1})^3 = 0, \tag{12}$$

because V is an antisymmetric matrix. To see this, note that

$$\mathrm{Tr}(dVV^{-1})^{2m+1} = (-1)^m \mathrm{Tr}(dVV^{-1})^{T2m+1}, \\ = (-1)^m \mathrm{Tr}(dVV^{-1})^{2m+1},$$

since transposing the matrices changes the ordering of the differential forms. This shows that $\mathrm{Tr}(dVV^{-1})^3$ vanishes, but not $\mathrm{Tr}(dVV^{-1})^5$. Thus the Wess-Zumino term does not contribute to anomalous Ward identities involving $U(1)_R$, which is consistent with the fact that the low-energy fermions correctly reproduce the anomalies of the microscopic theory which involves $U(1)_R$. The vanishing of Eq. (12) is related to $\pi_3(\mathcal{M}) = 0$ and the nonexistence of Skyrmions. Finally, note that the baryon number is part of SU(4), and the Wess-Zumino term does contribute to SU(4) currents.

The Wess-Zumino term in the presence of background SU(4) gauge fields A can be obtained from the result of Witten for QCD [2] (in the parity invariant form of Ref. [16]). It is conventionally written as

$$\Gamma(V) + \frac{1}{48\pi^2}Z(V,A),$$

where Z can be obtained from the one in Ref. [16] by the replacement

$$A_L \to A, \qquad A_R \to -A^T, \qquad \Sigma \to V, \qquad \Sigma^\dagger \to V^{-1}. \tag{13}$$

The variation under a gauge transformation is

$$\delta\left(\Gamma + \frac{1}{48\pi^2}Z\right) = -\frac{1}{24\pi^2}\int \mathrm{Tr}\epsilon d\left(AdA + \frac{1}{2}A^3\right) + \frac{1}{24\pi^2}\int \mathrm{Tr} - \epsilon^T d\left(A^T dA^T - \frac{1}{2}A^{T3}\right) \\ = -2\frac{1}{24\pi^2}\int \mathrm{Tr}\epsilon d\left(AdA + \frac{1}{2}A^3\right) \tag{14}$$

using the results of Ref. [2] and Eq. (13). Thus the Wess-Zumino functional contribution to the $SU(4)^3$ anomaly is twice that of a Weyl fermion in the fundamental representation of SU(4). The quarks in the microscopic theory are a **4** of SU(4), and contribute an anomaly of two [since they are a gauge SU(2) doublet]. The low-energy field fermions are a **6** of SU(4), which is a real representation, and so do not contribute to the anomaly. This determines the coefficient n in front of the Wess-Zumino term in the effective action to be $n = 1$. Note that Γ contributes to the anomalous Ward identity even though V transforms as a real representation of SU(4), and that $n = 1$ for two colors, unlike in QCD where $n = 2$.

At points on the moduli space where $V = J$,

$$J = \begin{pmatrix} i\sigma_2 & 0 \\ 0 & i\sigma_2 \end{pmatrix},$$

the flavor SU(4) group is broken into an Sp(4) subgroup. The $Sp(4)^3$ flavor anomaly matching condition is satisfied between the fermions in the high-energy and low-energy theories. One can check that Eq. (14) does not contribute to the anomalous Ward identity for Sp(4). For the Sp(4) subgroup, $J\epsilon J = \epsilon^T$, $JAJ = A^T$. Using these relations, it is easy to see that Eq. (14) vanishes. This must be the case, since Sp(4) does not have complex representations.

The above results are easily generalized to $Sp(2n-2)$ theories with $2n$ fundamentals, which also has a quantum deformed moduli space with a Pfaffian constraint [17]. The moduli space for supersymmetric QCD with $N = F = 2$ can be thought of as the complexification of SU(4)/Sp(4), which is the complexification of S^5. Similarly, the moduli space for the $Sp(2n-2)$ theories with $2n$ fundamentals is the complexification of $SU(2n)/Sp(2n)$, which has a deformation retract onto $SU(2n)/Sp(2n)$. It is known [2] that $\pi_2[SU(2n)/Sp(2n)] = \pi_3[SU(2n)/Sp(2n)] = 0$, but $\pi_5[SU(2n)/Sp(2n)] = \mathbb{Z}$, so the theory has a Wess-Zumino term but no Skyrmions.

The generalization to other quantum modified theories is more involved. In nonsupersymmetric theories with a flavor symmetry G broken into a subgroup H, the manifold of Goldstone boson fields is the compact manifold G/H. In this case, it has been proved, in general, that one can always construct a Wess-Zumino term that reproduces the correct anomalous Ward identities [18]. In supersymmetric theories, there are (noncompact) flat directions in addition to the usual G/H Goldstone boson directions. In general, the moduli space $\mathcal{M}$ is not a homogeneous space, and the unbroken flavor group can be different at different points of $\mathcal{M}$, as happens in supersymmetric QCD with $N = F > 2$. The Higgs mechanism can be used in this case to flow to supersymmetric QCD with $N = F = 2$, for which the Wess-Zumino term exists. This indicates that a Wess-Zumino term should also exist for $N = F > 2$, but it would be useful to have an explicit construction.

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