Ensembles of Singularities Generated by Surfaces with Polyhedral Symmetry

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The role of the symmetries in the topology of sets of Lagrangian singularities is studied in a simple physical model: the envelope of the rays emanating from a convex wave front invariant under the action of polyhedral groups. New point singularities are found of integer index and located at the vertices of the polyhedron or of its reciprocal. This remarkable layout results from the interplay between the symmetries of the singularities, the polyhedral symmetries, and the topology of the wave front. An application to fine-particle magnetic systems is given. [S0031-9007(98)06939-7]

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Singularities, as defined by the singularity theory, are encountered in physics under various aspects: caustics, shocks, critical points of stability diagrams, etc. [1]. One can extract from the singularities information on the physical system [2]. In general the singularities do not appear isolated, but they rather form ensembles (configurations) including strata of different dimensions, self-intersections, asymptotic branches, borders, and so on. The physical laws, the boundary conditions, or some external fields, often impose symmetries which may increase the complexity of the singular configuration. In the following we study precisely the relationship between the *symmetries* and the global *topology* of an ensemble of Lagrangian singularities (caustics [3]) defined as the focal set of a closed surface W. Two important features characterize these singularities.

On the one hand, each type of singularity possesses a local symmetry. This symmetry must be compatible with the symmetry imposed by the system; i.e., it is a subgroup of the (global) symmetry group. The compatibility condition raises an interesting problem when no generic (elementary) singularity can be compatible with one of the symmetries of the system. In this case, as we shall see, one may have instead new degenerated singularities, i.e., singularities composed of generic ones, so as to satisfy the imposed symmetry.

On the other hand, the caustic surface is not totally arbitrary, but it also obeys some topological rules. For instance, the number of the umbilical singularities is related to the Euler characteristic of the (compact) caustic [4,5]. Thus, any ensemble of singularities must obey simultaneously two constraints: the topological rules and the symmetry imposed by the physical system.

In this Letter we study, on an example, the interplay of these constraints on the configurations of Lagrangian singularities. We study the caustic defined as the envelope of the normals to a convex closed surface W invariant under the action of a polyhedral group of symmetries [discrete subgroup of the group $O(3)$]. We show that new singularities appear located in a remarkable dual way and that they result from the application of the two constraints due to the symmetries and the topology. The results are compared with some results obtained in the problem of the stability of magnetic grains [6].

Our model is a wave front propagating in a homogeneous medium of refractive index equal to 1. We define the initial wave front W with the required symmetry by introducing, as in the classical construction of the ellipse, a function V "sum of the distances to the foci": $W = \{(x,y,z), V(x,y,z) = \text{const}\}$. More precisely we fix n points P_i (the "foci") in $R^3 = \{x,y,z\}$ and n masses m_i. The function V is given by

$$V(P) = \sum_{i=1}^{n} m_i d(P, P_i), \tag{1}$$

where $d(P,P_i)$ denotes the distance from P to P_i. We assume now that $m_i = 1/n > 0$ for all i, so that the level surface W is closed. Moreover, we assume also that the P_i lie on the unit sphere and that they form the vertices of a regular polyhedron $\mathbb{P} = \{p,q\}$, in which q p-gons are surrounding each vertex. By construction, the wave front W, the family of the rays, which are the normals to W, and the caustic, which is the envelope of the rays, are left invariant under the action of the symmetry group of $\mathbb{P}$. We name the associated caustic a *polyhedral caustic.*

Since W is a convex surface enclosing the origin O, it is convenient to use the spherical coordinates r, α, β: $x = r\cos\alpha\cos\beta$, $y = r\cos\alpha\sin\beta$, $z = r\sin\alpha$. The equation of W: $V(P) = \text{const}$ defines implicitly r as a function of α and β. We denote by s the coordinate along the ray. A point $P = (x,y,z)$ of the congruence of the rays depends on α, β, and s through the relation

$$\begin{aligned} x(\alpha,\beta,s) &= r\cos\alpha\cos\beta + sV_x, \\ y(\alpha,\beta,s) &= r\cos\alpha\sin\beta + sV_y, \\ z(\alpha,\beta,s) &= r\sin\alpha + sV_z, \end{aligned} \tag{2}$$

where V_x stands for $\partial V/\partial x$, etc.

The relation (2) defines a mapping f from the source space $\{\alpha,\beta,s\}$ into the physical space $\{x,y,z\}$. The singular points are the points where f has a rank strictly less than the maximum possible value 3. They are given by the relation

$$\det \partial(x, y, z)/\partial(\alpha, \beta, s) = 0\,. \qquad (3)$$

The solution $s(\alpha, \beta)$ of this equation determines the singular set Σ. It cannot be found explicitly and its value is determined numerically, with relative errors of the order of 10^{-7}. Then the mapping f is applied to Σ to obtain the caustic $C = f(\Sigma)$. Equation (3) is a quadratic equation of s, showing that the caustic is composed of two sheets C_+ and C_-.

Here it is worth recalling that a caustic surface is never a regular surface. It possesses itself singular lines, the cusp lines A_3. These cusp lines have generally singular points: swallowtails A_4 or umbilics D_4. The umbilics are junction points between the two sheets C_+ and C_-. They may be classified in essentially two distinct ways. First, according to the number of cusp lines meeting at the umbilic point, they may be either of the elliptic type D_4^- (three cusps) or of the hyperbolic type D_4^+ (one cusp line). Second, they may have an index equal either to $+\frac{1}{2}$ or to $-\frac{1}{2}$ [4,7]. The elliptic umbilics have always an index equal to $-\frac{1}{2}$. The hyperbolic umbilics may have either an index equal to $+\frac{1}{2}$ (type "drop" D_4^{+d}) or an index equal to $-\frac{1}{2}$ (type "triangle" D_4^{+t}). In our simulations, we use the characterization of the index of an umbilic by the relative position of the ray with respect to the caustics [7].

Since $V(O) = 1$, the value of the constant defining the level surface W must be taken greater than 1, say equal to 2. The caustic surface is a compact (singular) surface without infinite branches. The caustic points are calculated by type, according to the method of the corank [8,9]. Figure 1(a) shows the caustic surface associated with a cube, each sheet being represented separately for a better understanding. The caustic is very intricate, because of the presence of many self-intersection lines. However, it is possible to reduce the complexity of the description by focusing our attention only on the singularities of smallest dimension, i.e., the point singularities, called *organizing centers* in the literature, because of their dominant role in the topology of the caustic [1].

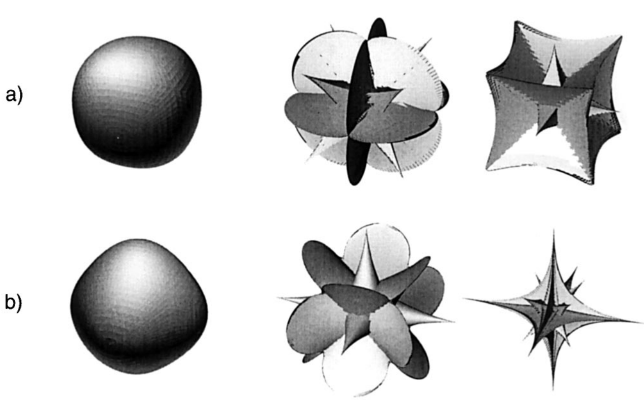

FIG. 1. Shape of the wave fronts W (left) constructed starting from (a) the cube, (b) the octahedron, and their associated caustics C. Both sheets of C are represented separately: C_+ (middle) and C_- (right).

This technique of identification of the singularities is now applied successively to the five polyhedral caustics. The results are summarized in Table I. Each column corresponds to a polyhedron $\mathbb{P}$. Each axis of symmetry is of type C_{nv} and passes through a vertex, a center of a face, or a center of an edge. It is denoted here by the value of n: 2, 3, 4, or 5. It is convenient to distinguish the two semiaxes forming an axis of symmetry, so that, for instance, there are four axes 3 passing through the vertices of a tetrahedron and four other axes 3 passing through the centers of its faces. All the axes 3 bear an elliptic umbilic D_4^- (see Table I). Their presence is expected, since they have locally the symmetry C_{3v}. The axes 4 and 5 bear nongeneric singularities, as expected too, since generic singularities do not have the symmetries C_{4v} or C_{5v}. In order to understand the nature of these new singularities, we slightly perturb W by changing the values of the masses. The nongeneric singularities split into pairs of hyperbolic umbilics: two D_4^{+d} (in the case of the octahedron, of the icosahedron, and of the dodecahedron) or two D_4^{+t} (in the case of the cube). These degenerate singularities are then noted by $2D_4^{+d}$ and $2D_4^{+t}$ in Table I. The sum of the indices of both singularities of each pair stands for their index g: $+1$ for the pair $2D_4^{+d}$, -1 for the pair $2D_4^{+t}$. The point singularities located on the axes of symmetry passing through the vertices form a second polyhedron $\mathbb{P}$. The point singularities located on the axes of symmetry passing through the centers of the faces form the reciprocal polyhedron of $\mathbb{P}$. In the case of the tetrahedron and of the cube, there exist also umbilics D_4^{+d} outside the axes of symmetry, in some planes of symmetry. We check that the total index of the umbilics is always equal to 2, i.e., the value of the Euler characteristic of W.

We note also that every D_4^- of each polyhedral caustic is surrounded by three symmetric butterflies A_5^1 [10] (not reported in Table I).

In Table I, the results concerning the icosahedron and its reciprocal, the dodecahedron, may be clearly put in correspondence by the reciprocation transforming vertices and faces. On the other hand, the results concerning the cube and its reciprocal, the octahedron, seem to be uncorrelated. The case of the tetrahedron gives no information on this issue, since it is self-reciprocal.

To explain this apparent asymmetry, we consider a general polyhedral caustic. The symmetry elements are those of some polyhedron $\{p, q\}$ [11]. Each element (axis or plane) bears singularities having a local symmetry compatible with the symmetry associated with it. We now assume that the number of the umbilics and the value of their indices are as small as possible. Then the axes 3 bear elliptic umbilics D_4^-. The axes 4 or

TABLE I. For each polyhedron $\{p, q\}$, the umbilics (degenerate or not) lie on symmetry elements which are the axes passing through the vertices (row "vertices"), the axes passing through the center of the faces (row "faces"), and some planes of symmetry (row "extra umbilics"). The symmetry of the axes is indicated by the value of n (see text). For each symmetry element, the table gives the nature, the number, and the index of the umbilics. It gives also (in the two first rows) the polyhedron formed by the umbilics. Only the caustics associated with the tetrahedron and the cube have extra umbilics. The last row indicates the sum of the indices of the umbilics.

	Tetrahedron $\{3,3\}$	Cube $\{4,3\}$	Octahedron $\{3,4\}$	Icosahedron $\{3,5\}$	Dodecahedron $\{5,3\}$
Vertices	3	3	4	5	3
Singularity	D_4^-	D_4^-	$2D_4^{+d}$	$2D_4^{+d}$	D_4^-
Number	4	8	6	12	20
Index	$-\frac{1}{2}$	$-\frac{1}{2}$	$+1$	$+1$	$-\frac{1}{2}$
Polyhedron	Tetrahedron	Cube	Octahedron	Icosahedron	Dodecahedron
Faces	3	4	3	3	5
Singularity	D_4^-	$2D_4^{+t}$	D_4^-	D_4^-	$2D_4^{+d}$
Number	4	6	8	20	12
Index	$-\frac{1}{2}$	-1	$-\frac{1}{2}$	$-\frac{1}{2}$	$+1$
Polyhedron	Tetrahedron	Octahedron	Cube	Dodecahedron	Icosahedron
Extra umbilics					
Singularity	D_4^{+d}	D_4^{+d}	…	…	…
Number	12	24			
Index	$+\frac{1}{2}$	$+\frac{1}{2}$			
Total index	2	2	2	2	2

5 are incompatible with generic singularities, and they bear degenerate singularities with an index $g = 0$ or ± 1 (the most simple cases). Of course, other umbilics may be present. The number N of such "extra" umbilics is equal to half of the number of the triangles of the $\{p, q\}$ tiling of the sphere, i.e., $N = 4kpq/[4 - (p - 2)(q - 2)]$, where $k = 0$ or 1. We now write that the total index is equal to the Euler characteristic of W, i.e., 2. In the case of the tetrahedral symmetry, we obtain the relation $\pm 6k = 6$ (the sign in the left hand member refers to that of the index of the extra umbilics). We then obtain the solution $k = 1$, i.e., the solution for the tetrahedron: 12 extra umbilics D_4^{+d}. For the cubic symmetry, we obtain the relation $6g \pm 12k = 6$ which admits the two solutions $g = 1$, $k = 0$ and $g = -1$, $k = 1$. The first one corresponds to the solution for the octahedron: degenerate singularities of index 1 and no extra umbilic. The second one corresponds to the solution for the cube: degenerate singularities of index -1 and 24 extra umbilics D_4^{+d}. Finally, for the icosahedral symmetry, we obtain the relation $12g \pm 30k = 12$ which admits the unique solution $g = 1$, $k = 0$. It corresponds to the solution for the icosahedron and for the dodecahedron: degenerate singularities of index $+1$ and no extra umbilics. Moreover, we can specify the nature of the umbilics, since the value $g = +1$ corresponds to the pair $2D_4^{+d}$. The value $g = -1$, found in the cubic case, gives *a priori* two possibilities: $2D_4^{+t}$ (the solution of the model) and $2D_4^-$. In fact, the latter solution can be obtained from the former one through a *symmetry-preserving* bifurcation by decreasing the value of the constant of the level surface W. Both cases are two realizations of a unique case. So we conclude that all the information contained in Table I is recovered on the basis of the compatibility between the local symmetries of the singularities and the imposed symmetries (implying that D_4^- lie on the axes 3, and so on) and on the basis of a minimal index for the degenerate singularities ($g = 0$ or ± 1). There exist four elementary polyhedral caustics, and they are provided by our simple model. The duality between the cases of the icosahedron and of the dodecahedron simply expresses that they form a unique case. We note that the number of the basic polyhedral caustics (4) is neither equal to the number of polyhedra (5) nor equal to the number of polyhedral symmetries (3).

Although our model is an optical one, the results apply as well to other (Lagrangian) symmetrical singularities. This circumstance allows us to compare them with those obtained in a recent study of the stability diagram of a magnetic nanoparticle placed in a magnetic field $\mathbf{H}$ [6]. In this example the coordinates H_x, H_y, H_z play the role of our physical coordinates x, y, z. The magnetic energy depends on the magnetization orientation $\mathbf{m}$ and on the applied field $\mathbf{H}$ [12]. The equilibrium conditions define E as a function of $\mathbf{H}$ and the level surface $E(\mathbf{H}) = \text{const}$ is a convex closed surface W_{mag}. Its caustic represents the fields for which the magnetization orientation undergoes a jump (switching fields) and may be experimentally determined [6]. In the case of the cubic symmetry the caustic of the switching fields exactly presents the topology of our caustic for the octahedron [see Fig. 1(b)]. In particular, it presents the six degenerate umbilics around which the caustic surface is locally conic. Then without any calculation, we are able to give the nature and the indices of all the umbilic points of this caustic (see column "octahedron" of Table I). We also predict how the degenerate singularities split under the effect of a small symmetry breaking of the magnetic system. It would be very interesting to check experimentally this point. It would be also interesting to know if the second solution for the cubic symmetry (column "cube" of Table I) may be realized in this example by changing the value of some parameter preserving the cubic symmetry.

Other applications of our results may include as well phase transitions with symmetries [13], ballistic heat pulses in crystals [14], and bound states of Hamiltonian systems [15].

Finally, we point out that our model is also well adapted to the study of the dihedral caustics and to attack the open problem of the caustic bifurcations preserving (totally or partially) imposed symmetries (cf., for instance, the symmetry-preserving bifurcation in the discussion of our cubic case).

In conclusion, we have shown that there exist four elementary polyhedral caustics. We have described the nature and the layout of their point singularities. Except for the tetrahedral symmetry, the caustic contains degenerate umbilics of integer index ± 1. We stress that these characteristics result from the conditions simultaneously imposed by the *global* (polyhedral) symmetries of the problem, by the *local* symmetries associated with the singularities, and by the topology of the initial wave front.

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