PHYSICAL REVIEW LETTERS

VOLUME 81 | 17 AUGUST 1998 | NUMBER 7

Minimal Optimal Generalized Quantum Measurements

J. I. Latorre, P. Pascual, and R. Tarrach

Departament d'Estructura i Constituents de la Matèria, Facultat de Física, Universitat de Barcelona, Diagonal 647, E-08028 Barcelona, Spain

(Received 26 March 1998)

Optimal and finite positive operator valued measurements on a finite number N of identically prepared systems have recently been presented. With physical realization in mind, we propose here optimal and *minimal* generalized quantum measurements for two-level systems. We explicitly construct them up to $N = 7$ and verify that they are minimal up to $N = 5$. [S0031-9007(98)06751-9]

PACS numbers: 03.65.Bz, 03.67.–a

Measurements disclose unknown information. They should disclose as much information as possible by using the least amount of physical resources. We present here, for the first time, the most efficient measurements for the simplest quantum systems.

Consider a spin-$\frac{1}{2}$ particle (or any other two-level system) which is in a pure state $|\Psi\rangle$ about which we do not know anything, that is, its spin points with equal probability into any direction. By performing a measurement on the system, one learns something about $|\Psi\rangle$, that is, the *a priori* uniform probability distribution becomes *a posteriori* a nonuniform distribution. Suppose now we have N identical copies of $|\Psi\rangle$, $|\Psi\rangle^N \equiv |\Psi\rangle \otimes |\Psi\rangle \otimes |\Psi\rangle \cdots \otimes |\Psi\rangle$ (N times). Measurements on this enlarged system allow one to learn more about $|\Psi\rangle$. The amount of knowledge that measurements allow one to extract from $|\Psi\rangle^N$ about $|\Psi\rangle$ is a monotonically increasing function of N. Only in the limit $N \to \infty$ can $|\Psi\rangle$ be determined exactly. This is because only in this limit are $|\Psi\rangle^N$ and $|\Psi'\rangle^N$ orthogonal whenever $|\Psi\rangle \neq |\Psi'\rangle$, and thus distinguishable by an adequate measurement.

For finite N, Massar and Popescu [1] (see also Holevo [2]) obtained the *optimal* measurement procedure for spin-$\frac{1}{2}$ particles. Their procedure, leading to the maximal knowledge about $|\Psi\rangle$, corresponds to a positive operator valued measurement (POVM) consisting of an *infinite* isotropic set of projectors in the Hilbert space of $|\Psi\rangle^N$. It is a measurement on the *combined* system. By Neumark's theorem [3,4] this corresponds to a von Neumann measurement in an infinitely dimensional extension of the Hilbert space of $|\Psi\rangle^N$. This makes the procedure academic, since it cannot be realized physically.

The next step was taken by Derka, Buzek, and Ekert [5]. They explicitly construct an optimal *finite* POVM, thus making the procedure, in principle, accessible to the laboratory, and thus of relevance to quantum computation and quantum communication. They quantify the acquired knowledge about $|\Psi\rangle$ by the mean fidelity, $\overline{f}$, whose maximal value obtained by their procedure is

$$\overline{f}_{\max} = \frac{N+1}{N+2}. \tag{1}$$

Their POVM requires a finite number $n = (N+1)^2$ of projectors in the Hilbert space of $|\Psi\rangle^N$. It is thus an optimal, finite, generalized quantum measurement. But it is not minimal: Optimal POVMs with a smaller number of projectors exist, as we will show. They allow one to learn the same by reading a smaller output. When it comes to physical realizations this should be an advantage.

Here we present explicit results on optimal, finite, and, futhermore, minimal POVMs. The number of projectors n they require is roughly one-third the number needed by the only optimal and finite measurements known up to now [5]. We have proceeded from $N = 2$ up to $N = 5$ case-by-case, because we do not know how to build the POVM algorithmically. They are optimal and minimal. Then we construct optimal POVMs for $N = 6$

and $N = 7$ which we strongly believe to be minimal. This belief is based on a bit of mathematical intuition and some numerical frustration, but we have not been able to rigorously exclude POVMs with one projector less. We finally propose and explain a formula which gives the minimal n as a function of N and which reproduces all of our explicit results.

Let us first introduce some notation (we will try to follow Ref. [5] whenever possible). Our POVM is given by a finite set of n one-dimensional projectors built from the states of maximal spin, $s = N/2$, and maximal spin component in some direction, $|\theta_r, \psi_r\rangle^N$, $r = 1, \ldots, n$, where $\vec{\sigma} \cdot \hat{n}(r)\,|\theta_r, \psi_r\rangle$, $\hat{n}(r) = (\sin\theta_r \cos\psi_r, \sin\theta_r \sin\psi_r, \cos\theta_r)$ and such that

$$\sum_{r=1}^{n} c_r^2 |\theta_r, \psi_r\rangle^{NN}\langle\theta_r, \psi_r| = I^{(s=N/2)}, \qquad 0 < c_r^2 \le 1\,. \tag{2}$$

Here the right-hand side represents the identity in the maximal spin space. Notice that n has to be larger than the dimension of the maximal spin space, $N + 1$, as $n = N + 1$ would require the n projectors of Eq. (2) to be orthogonal, which they are not. The extension of Eq. (2) to the complete 2^N-dimensional Hilbert space is straightforward, but irrelevant, as the corresponding projectors, being orthogonal to $|\Psi\rangle^N$, do not allow us to increase our knowledge about $|\Psi\rangle$.

We know from Refs. [1,2,5] that a POVM of the type we are considering is optimal. This means that the mean fidelity,

$$\bar{f} \equiv \sum_{r=1}^{n} \frac{1}{4\pi} \int_{-1}^{1} d\cos\theta \times \int_0^{2\pi} d\psi |^N\langle\Psi\,|\,\theta_r, \psi_r\rangle^N|^2 c_r^2 |\langle\Psi\,|\,\theta_r, \psi_r\rangle|^2, \tag{3}$$

where $|\Psi\rangle \equiv |\theta, \psi\rangle = \vec{\sigma} \cdot \hat{n}|\theta, \psi\rangle$, $\hat{n} = (\sin\theta \cos\psi, \sin\theta \sin\psi, \cos\theta)$. It was also shown in Ref. [5] that for optimal POVMs Eq. (2) can be substituted by the much simpler one,

$$\sum_{r=1}^{n} c_r^2 |^N\langle\theta, \psi\,|\,\theta_r, \psi_r\rangle^N|^2 = 1, \qquad \forall|\theta, \psi\rangle\,. \tag{4}$$

This is therefore the equation we want to study and solve, i.e., find c_r^2, θ_r and ψ_r, $r = 1, 2, \ldots, n$, for the smallest n possible.

It is not difficult to prove from the explicit expression for $|^N\langle\theta, \psi\,|\,\theta_r, \psi_r\rangle^N|^2$ and expanding monomials in terms of Legendre polynomials that Eq. (4) is equivalent to

$$\begin{gathered} \sum_{r=1}^{n} c_r^2 = N + 1\,, \\ \sum_{r=1}^{n} c_r^2 P_l^M(\cos\theta_r) e^{iM\psi_r} = 0\,, \\ L = 1, \ldots, N, \qquad M = 0, \ldots, L\,, \end{gathered} \tag{5}$$

where the dependence on θ, ψ has been traded for a set of equations. Again, after some algebra, this set of equations can be shown to be equivalent to

$$\begin{gathered} \sum_{r=1}^{n} c_r^2 = N + 1\,, \\ \sum_{r=1}^{n} c_r^2 n_\alpha(r) = 0\,, \\ \sum_{r=1}^{n} c_r^2 n_\alpha(r) n_\beta(r) = \frac{N + 1}{3}\,\delta_{\alpha\beta}\,, \\ \sum_{r=1}^{n} c_r^2 n_\alpha(r) n_\beta(r) n_\gamma(r) = 0\,, \\ \vdots \end{gathered} \tag{6}$$

which, in compact form, reads

$$\sum_{r=1}^{n} c_r^2 \hat{n}(r)^q = \frac{1 + (-1)^q}{2} \frac{N + 1}{q + 1} I^{(q)}, \qquad q = 0, \ldots, N\,, \tag{7}$$

where $\hat{n}(r)^q \equiv \hat{n}(r) \otimes \hat{n}(r) \otimes \cdots \otimes \hat{n}(r)$ with q factors, and $I^{(q)}$ is the invariant symmetric rank q tensor, trace normalized to $q + 1$, $I^{(0)} \equiv 1$, $I^{(2)}_{\alpha\beta} \equiv \delta_{\alpha\beta}$, $I^{(4)}_{\alpha\beta\gamma\delta} \equiv \frac{1}{3}(\delta_{\alpha\beta}\delta_{\gamma\delta} + \delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma})$, etc. In order to simplify our future discussion we also note that Eq. (7) can be contracted with $\hat{n}(i)^q$ leading to

$$\sum_{r \neq i}^{n} c_r^2 [\hat{n}(r) \cdot \hat{n}(i)]^q = \frac{1 + (-1)^q}{2} \frac{N + 1}{q + 1} - c_i^2, \qquad i = 1, \ldots, n, \qquad q = 0, \ldots N\,. \tag{8}$$

Let us pause and reflect on the meaning of the above set of equations. As N increases, more equations in the hierarchy of Eq. (6) must be verified forcing the distribution of c_r^2 and $\hat{n}(r)$ to approach the form of a continuous uniform angular distribution. Thus, for finite N, we do expect to obtain highly symmetric solutions. No algorithm to find the minimal n which produces a solution of the truncated set of equations has emerged from our efforts. We have, therefore, proceeded case-by-case from $N = 2$ upwards.

Let us discuss in some detail the deduction of the explicit solution in the case $N = 2$. We have to solve the first three sets of equations in Eq. (6) for the minimal possible n. Using Eq. (8) the manifestly non-negative combination

$$\begin{aligned} S &\equiv \sum_{r \neq i}^{n} c_r^2 [b_i + \hat{n}(i) \cdot \hat{n}(r)]^2 \\ &= b_i^2(3 - c_i^2) - 2b_i c_i^2 + 1 - c_i^2 \ge 0\,, \\ &\forall i = 1, \ldots, n \end{aligned} \tag{9}$$

can be evaluated. It reaches its minimum for

$$b_i = \frac{c_i^2}{3 - c_i^2} \Rightarrow S = \frac{3 - 4c_i^2}{3 - c_i^2} \ge 0\,. \tag{10}$$

This forces $c_i^2 = \frac{3}{4}$ and, furthermore,

$$\sum_{i=1}^{n} (3 - 4c_i^2) = 3(n - 4) \geq 0\,, \tag{11}$$

proving that $n \geq 4$. It is easy to see that a solution that saturates the bound exists. Indeed, taking the largest possible value for all c_i^2, that is, $c_i^2 = \frac{3}{4}$, in our original expression for S, we get

$$S = \frac{3}{4} \sum_{r \neq i}^{n} \left(\frac{1}{3} + \hat{n}(i)\hat{n}(r) \right)^2 = 0\,, \tag{12}$$

which implies that every term in the sum must vanish and leads to the final result,

$$\begin{aligned} n_{\min}(N = 2) &= 4\,, \\ c_i^2 = \tfrac{3}{4}, &\qquad i = 1, \ldots, 4\,, \\ \hat{n}(i) \cdot \hat{n}(j) = -\tfrac{1}{3}, &\qquad \forall i \neq j\,. \end{aligned} \tag{13}$$

This solution corresponds to a regular tetrahedron. The minimal optimal POVM for $N = 2$ is thus organized as a platonic polyhedron, c_i^2 playing the role of the distance to the vertices from the center and $\hat{n}(i)$ pointing into the directions of the vertices. As anticipated, this solution is unique by construction and stands as the smallest discretization of angular integration.

The key idea to find out the above solution was to select a manifestly positive combination of all of the equations needed at level N. Let us take advantage of this clue in the case $N = 3$, which corresponds to solving the first four sets of equations in Eq. (6). We combine them into the, again, manifestly non-negative expression

$$\begin{aligned} S &\equiv \sum_{r \neq i}^{n} c_r^2 [1 + \hat{n}(i) \cdot \hat{n}(r)][b_i + \hat{n}(i) \cdot \hat{n}(r)]^2 \\ &= b_i^2 (4 - 2c_i^2) + 2b_i \left(\frac{4}{3} - 2c_i^2 \right) + \left(\frac{4}{3} - 2c_i^2 \right) \\ &\geq 0, \qquad \forall i = 1, \ldots, n\,. \end{aligned} \tag{14}$$

The minimum of S corresponds to

$$b_i = -\frac{1}{3} \frac{2 - 3c_i^2}{2 - c_i^2} \Rightarrow S = \frac{8}{9} \frac{2 - 3c_i^2}{2 - c_i^2}\,. \tag{15}$$

We, thus, deduce that all $c_i^2 \leq \frac{2}{3}$, and

$$\sum_{i=1}^{n} (2 - 3c_i^2) = 2(n - 6) \geq 0\,. \tag{16}$$

The bound is then $n \geq 6$. A solution that saturates the bound exists and can be found by setting all $c_i^2 = \frac{2}{3}$, leading to

$$S = \sum_{r \neq i} c_r^2 [1 + \hat{n}(i) \cdot \hat{n}(r)][\hat{n}(i) \cdot \hat{n}(r)]^2 = 0\,. \tag{17}$$

Every term in the sum must vanish; thus, the scalar products of any pair of vectors, $\hat{n}(i) \cdot \hat{n}(r)$, are constrained to be either 0 or -1. It is easy to use Eq. (6) to show that

$$\begin{aligned} n_{\min}(N = 3) = 6, &\qquad c_i^2 = \tfrac{2}{3}, \qquad i = 1, \ldots, 6\,, \\ \hat{n}(i) \cdot \hat{n}(j) = 0, &\qquad \forall i \neq j\,, \end{aligned} \tag{18}$$

except $\hat{n}(1) \cdot \hat{n}(6) = \hat{n}(2) \cdot \hat{n}(4) = \hat{n}(3) \cdot \hat{n}(5) = -1$.

This solution corresponds to a regular octahedron. Once again a platonic polyhedron underlies the unique, optimal, and minimal POVM for $N = 3$.

For $N = 4$ we have found it convenient to start from

$$\sum_{r \neq i}^{n} c_r^2 \{ b_i + d_i \hat{n}(i) \cdot \hat{n}(r) + [\hat{n}(i) \cdot \hat{n}(r)]^2 \}^2 \geq 0\,. \tag{19}$$

Minimization with respect to b_i and d_i eventually leads to

$$(\tfrac{5}{4} - c_i^2)(\tfrac{5}{9} - c_i^2) \geq 0\,. \tag{20}$$

and

$$\sum_{i=1}^{n} \left(\frac{5}{9} - c_i^2 \right) = 5(n - 9) \geq 0\,, \tag{21}$$

which implies $n \geq 9$. For $n = 9$, the values obtained for c_i^2, $c_i^2 = \frac{5}{9}$, and $\hat{n}_i \cdot \hat{n}_r$, from saturating the bound, do not satisfy Eq. (6). Thus $n > 9$ strictly. Analyzing more elaborated bounds, we have been able to prove that, for $n = 10$, the c_i^2 cannot all be identical. By means of numerical inspiration, we have found an explicit solution for $n = 10$. Two of the c_i^2 turn out to be equal and smaller than the rest, which are also equal among them, and the $\hat{n}(i)$ point to the vertices of a figure made as a twisted prism with pyramidal caps (its explicit form is given below in Table I). We have therefore encountered a somewhat irregular but minimal solution to the POVM in the $N = 4$ case. The *modus operandi* is always related to exploiting a manifestly non-negative combination of all of the equations to be solved.

For $N = 5$ our starting point is

$$\sum_{r \neq i}^{n} c_r^2 (1 + \hat{n}_i \cdot \hat{n}_r)[b_i + d_i \hat{n}_i \cdot \hat{n}_r + (\hat{n}_i \cdot \hat{n}_r)^2]^2 \geq 0\,, \tag{22}$$

TABLE I. Minimal optimal POVMs for $N = 2$, 3, 4, 5.

N	r	c_r^2	$\cos\theta_r$	$\frac{1}{\pi}\psi_r$
2	1	$\frac{3}{4}$	1	0
	2−4		$-\frac{1}{3}$	$\frac{2}{3}(r-2)$
3	1		1	0
	2	$\frac{2}{3}$	−1	0
	3−6		0	$\frac{1}{2}(r-3)$
4	1	$\frac{5}{12}$	1	0
	2		−1	0
	3−6	$\frac{25}{48}$	$\frac{1}{\sqrt{5}}$	$\frac{1}{2}(r-3)$
	7−10		$-\frac{1}{\sqrt{5}}$	$\frac{1}{2}(r-\frac{13}{2})$
5	1		1	0
	2	$\frac{1}{2}$	−1	0
	3−7		$\frac{1}{\sqrt{5}}$	$\frac{2}{5}(r-3)$
	8−12		$-\frac{1}{\sqrt{5}}$	$\frac{2}{5}(r-\frac{15}{2})$

which, after minimization, leads to

$$\left(c_i^2 - \frac{1}{2}\right) \geq 0 \Rightarrow \sum_{i=1}^{n} (1 - 2c_i^2) = n - 12 \geq 0 \,. \quad (23)$$

Thus $n \geq 12$. For $n = 12$ we obtain a solution that does saturate the bound (in analogy to $N = 2, 3$). The explicit, unique, minimal solution is made with all $c_i^2 = \frac{1}{2}$ and $\hat{n}(i) \cdot \hat{n}(j) = -1,\ 1/\sqrt{5},\ -1/\sqrt{5}$, which corresponds to an icosahedron. Again, we defer the detailed structure of the solution to Table II.

Starting from expressions such as Eqs. (19) and (22), but with a cubic instead of quadratic polynomial, one can prove that $n > 16$ and $n > 20$ for $N = 6$ and 7, respectively. Exhaustion has prevented us from filling the gap between these lower bounds and the solutions with $n = 18$ and $n = 22$, respectively, which we have been able to build explicitly. Notice that, of the four cases, $N = 2$, 3, 4, and 5, for which we give a complete proof, for three of them, all but $N = 4$, our solution is also unique and corresponds to constant c_r^2.

TABLE II. Optimal POVMs for $N = 6, 7$.

N	r	c_r^2	$\cos\theta_r$	$\frac{1}{\pi}\,\psi_r$
6	1 2	$\frac{14}{45}$	± 1	0
	3–6 7–10	$\frac{7(410+\sqrt{30})}{7200}$	$\pm\frac{\sqrt{13+2\sqrt{30}}}{7}$	$\frac{1}{2}(r-3)$ $\frac{1}{2}(r-\frac{13}{2})$
	11–14 15–18	$\frac{7(410-\sqrt{30})}{7200}$	$\mp\frac{\sqrt{13+2\sqrt{30}}}{7}$	$\frac{1}{2}(r-11)$ $\frac{1}{2}(r-\frac{29}{2})$
7	1 2	$\frac{10}{27}$	± 1	0
	3–7 8–12	$\frac{147+\sqrt{105}}{405}$	$\pm\frac{1}{2}\sqrt{1 + 3\sqrt{\frac{3}{35}}}$	$\frac{2}{5}(r-3)$ $\frac{2}{5}(r-\frac{15}{2})$
	13–17 18–22	$\frac{147-\sqrt{105}}{405}$	$\mp\frac{1}{2}\sqrt{1 + 3\sqrt{\frac{3}{35}}}$	$\frac{2}{5}(r-13)$ $\frac{2}{5}(r-\frac{35}{2})$

We have summarized all of our result in the two tables. We have also checked that they all satisfy the equations for optimal POVMs of Ref. [5]. Having in our hands all of these concrete solutions, it is possible to speculate on which $n_{\min}$ corresponds to a given N. The formula we propose is

$$n_{\min}(N) = \min\left(1 + \left[\frac{2 + (N+1)^2}{3}\right],\quad 4 + 2\left[\frac{N}{2}\right] + 2\left[\frac{2}{3}\left[\frac{N}{2}\right]^2\right]\right), \quad (24)$$

where square brackets mean integer part. To justify it, let us first note that the number of independent equations in Eq. (5) or (7) is $(N+1)^2$. The number of unknown variables in these equations is $3n - 3$, where rotation invariance has been used to fix $\theta_1 = \psi_1 = \psi_2 = 0$. Let us clearly state that the problem of finding rigorously the minimal n, which for each N allows one to solve the nonlinear system of Eq. (6), is beyond our mathematical skills. However, the explicit cases $N = 2$ to 7 seem to suggest that for this system one can always find a solution when the number of unknown variables is at least equal to the number of equations,

$$3n - 3 \geq (N+1)^2. \quad (25)$$

The minimal n satisfying Eq. (25) leads to the first expression in Eq. (2). On the other hand, limiting ourselves to solutions with even n and for which $\hat{n}_r + \hat{n}_{r-1} = 0$, $c_r^2 = c_{r-1}^2$, $r = 2, 4, \ldots, n$, the system of Eq. (6) reduces then to its even q part. The assumption that the number of variables is at least the number of equations,

$$\frac{3n}{2} - 3 \geq 1 + 3\left[\frac{N}{2}\right] + 2\left[\frac{N}{2}\right]^2, \quad (26)$$

now leads to a minimal even n given by the second expression in Eq. (24). This is the justification of Eq. (24). It gives $n_{\min}(6) = 18$ and $n_{\min}(7) = 22$, which corresponds precisely to the minimal solutions which we have been able to construct.

This means that one can do with roughly one-third the number of projectors required by the procedure of Ref. [5]. It turns out that for N even the minimum is the first expression and for N odd the second. Also $n_{\min}$ is always even.

Let us wind up by noting that we have used here the mean fidelity as a measure of acquired knowledge, but we could have used the more information-theoretic decrease in Shannon entropy, as, e.g., done in a related problem by Peres and Wootters [6]. Our conclusion would have been the same: We would have built the same optimal, minimal POVMs.

Financial support from CICYT, Contract No. AEN95-0590, and from CIRIT, Contract No. 1996GR00066, is acknowledged.

[1] S. Massar and S. Popescu, Phys. Rev. Lett. **74**, 1259 (1995).
[2] A. S. Holevo, *Probabilistic and Statistical Aspects of Quantum Theory* (North-Holland, Amsterdam, 1982).
[3] M. A. Neumark, C.R. Acad. Sci. URSS **41**, 359 (1943).
[4] A. Peres, Found. Phys. **20**, 1441 (1990).
[5] R. Derka, V. Buzek, and A. K. Ekert, Phys. Rev. Lett. **80**, 1571 (1998).
[6] A. Peres and W. K. Wootters, Phys. Rev. Lett. **66**, 1119 (1991).