Hamiltonian Time Evolution for General Relativity

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Hamiltonian time evolution in terms of an explicit parameter time is derived for general relativity, even when the constraints are not satisfied, from the Arnowitt-Deser-Misner-Teitelboim-Ashtekar action in which the slicing density $\alpha(x,t)$ is freely specified while the lapse $N = \alpha g^{1/2}$ is not. The constraint "algebra" becomes a well-posed evolution system for the constraints; this system is the twice-contracted Bianchi identity when $R_{ij} = 0$. The Hamiltonian constraint is an initial value constraint which determines $g^{1/2}$ and hence N, given α. [S0031-9007(98)06792-1]

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A minor change [1,2] in the Arnowitt-Deser-Misner (ADM) action principle [3,4] leads to striking consequences for the understanding of general relativity in Hamiltonian form. Recent work on hyperbolic formulations [5–11] of general relativity indicates [9,11] that the freely specifiable quantity that determines the slicing of spacetime is not the lapse N but the "slicing density" $\alpha(x,t) = Ng^{-1/2}$ (where $g = \det g_{ij}$ is the determinant of the spatial metric). Altering the action principle to take this into account leads to a number of key results: (1) equations of motion that are equivalent to $R_{ij} = 0$, not $G_{ij} = 0$, (2) a Hamiltonian vector field that generates time evolution even when the constraints are not satisfied, (3) a constraint algebra that is a homogeneous symmetric hyperbolic system that dynamically preserves the constraints, and (4) a new understanding of the Dirac algebra.

Before beginning, let us fix notation and state a few elementary results. We work on a manifold $\Sigma \times R$ with a "foliation-adapted" co-basis for the metric

$$ds^2 = -N^2(dt)^2 + g_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt). \tag{1}$$

Here, N is the lapse function (a space scalar) and β^i is the spatial shift vector. Overbars denote spatial quantities, in particular $\bar{R}$ the spatial curvature scalar obtained from g_{ij} and $\bar{\nabla}_i$ the corresponding spatial covariant derivative. We also introduce the extrinsic curvature K_{ij} and its trace $H \equiv K_k{}^k$. The momentum conjugate to the metric is a density of weight one, $\pi^{ij} = g^{1/2}(Hg^{ij} - K^{ij})$. The natural time derivative for evolution $\hat{\partial}_0$ acts in the normal future direction to the spacelike slice and is denoted by an overdot. It is given by $\hat{\partial}_0 = \partial_t - \mathcal{L}_{\boldsymbol{\beta}}$, where $\mathcal{L}_{\boldsymbol{\beta}}$ is the Lie derivative along the shift $\boldsymbol{\beta}$.

We have found in work on hyperbolic formulations of the equations of evolution of general relativity which have no unphysical characteristics [8–15] that we must, in essence, use the Choquet-Bruhat "algebraic gauge" [5] to restrict the ordinary lapse N. The weight-minus-one lapse (the slicing density) $\alpha = Ng^{-1/2} = \alpha(x,t)$ is *freely* specifiable while N is not [16]. (The slicing density α is also used prominently in [1,2,17].) Indeed, if one computes $\hat{\partial}_0 \log\alpha = f(x,t)$ from a given $\alpha(x,t)$, then one finds

$$g^{1/2}\hat{\partial}_0\alpha = \hat{\partial}_0 N + N^2 H = Nf, \tag{2}$$

the equation of harmonic time slicing [8,12] with $f(x,t) = \hat{\partial}_0 \log\alpha$ acting as a "gauge source" [18]. Combining this with the 3 + 1 equation for $\dot{H}$ from the trace of (10) below, one obtains a quasilinear wave equation for N. Every foliation is described by such a wave equation for some value of α. This wave equation for N [8,9,15] played a vital role in the development of our first order symmetric hyperbolic "Einstein-Ricci" system [8,10–12] and reflects the built-in *causality* which comes from working with α. We conclude that N, which determines the proper time as $N\delta t$ between slices $t = t'$ and $t = t' + \delta t$, is a *dynamical variable* (cf. [17]), closely connected to $g^{1/2}$. N can also be seen to be determined from α and the Hamiltonian constraint, the latter written as the generalized and completed conformal "Lichnerowicz equation" [19–22]; the explicit form in [21] is the suitable one for present purposes. From this perspective, the Hamiltonian constraint plays its familiar role as an initial value constraint which determines $g^{1/2}$ given a complete set of freely specified data [20]. The important insight is that this in turn determines N from α, so that the Hamiltonian constraint does not fix the time but does fix the rate of proper time τ with respect to t: $d\tau/dt = \alpha g^{1/2} = N$ along the normal ∂_0.

Motivated by these findings, we alter the undetermined multiplier in the ADM canonical action principle from N to α. Using α has the effect of altering the Hamiltonian density from $\mathcal{H}$ to

$$\tilde{\mathcal{H}} = g^{1/2}\mathcal{H} = \pi^{ij}\pi_{ij} - \tfrac{1}{2}\pi^2 - g\bar{R}, \tag{3}$$

the latter being of scalar weight plus two and a rational function of the metric. (Note, we reserve the phrase "Hamiltonian constraint" to refer to the equation $\tilde{\mathcal{H}} = 0$ and use "Hamiltonian density" for $\tilde{\mathcal{H}}$, which may not vanish, similarly for the momentum constraint and density.) This leads to Teitelboim's [1] and Ashtekar's [2] modification of the ADM action ($16\pi G = 1 = c$)

$$S_{\mathrm{ADMTA}}[\mathbf{g},\pi;\alpha,\boldsymbol{\beta}] = \int d^4x(\pi^{ij}\dot{g}_{ij} - \alpha\tilde{\mathcal{H}}), \tag{4}$$

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where we use Kuchař's notation indicating functional and explicit function dependence. Boundary terms are ignored here as they are not the focus of the present analysis. [There are no difficulties in obtaining, for example, the ADM energy surface integral. One requires the same asymptotic conditions as always, maintaining $N \to 1 + O(r^{-1})$—not $\alpha \to 1 + O(r^{-1})$—while recalling $N = g^{1/2}\alpha$.] The vacuum case is considered, but to add minimally coupled matter and/or a cosmological constant is straightforward. The momentum density $\mathcal{H}_i$ has been absorbed into $\pi^{ij}\dot{g}_{ij}$ through use of the time derivative $\hat{\partial}_0$. Explicitly, the Lie derivative term in $\dot{\pi}^{ij}$ is, up to a divergence,

$$2\beta^i \bar{\nabla}_j \pi_i{}^j = -\beta^i \mathcal{H}_i \,. \tag{5}$$

Consider a general variation of the modified Hamiltonian density

$$\begin{aligned}\delta\tilde{\mathcal{H}} &= (2\pi_{ij} - g_{ij}\pi)\delta\pi^{ij} \\ &\quad + (2\pi^{ik}\pi_k{}^j - \pi\pi^{ij} + g\bar{R}^{ij} - gg^{ij}\bar{R})\delta g_{ij} \\ &\quad - g(\bar{\nabla}^i\bar{\nabla}^j\delta g_{ij} - g^{ij}\bar{\nabla}^k\bar{\nabla}_k\delta g_{ij})\,. \end{aligned} \tag{6}$$

Note that (6) does not involve either the Hamiltonian or momentum densities while, in contrast, the variation of the ADM Hamiltonian density $\delta\mathcal{H} = \delta(g^{-1/2}\tilde{\mathcal{H}})$ does contain a term proportional to the Hamiltonian density.

Requiring that $S_{\rm ADMTA}$ be stationary under a variation with respect to π^{ij} gives the definition of the extrinsic curvature

$$\dot{g}_{ij} = \alpha\,\frac{\delta\tilde{\mathcal{H}}}{\delta\pi^{ij}} = \alpha(2\pi_{ij} - g_{ij}\pi) \equiv -2NK_{ij}\,. \tag{7}$$

Requiring that it be stationary under a variation with respect to g_{ij} gives the equation of motion

$$\begin{aligned}\dot{\pi}^{ij} &= -\alpha\,\frac{\delta\tilde{\mathcal{H}}}{\delta g_{ij}} \\ &= -\alpha g(\bar{R}^{ij} - g^{ij}\bar{R}) - \alpha(2\pi^{ik}\pi_k{}^j - \pi\pi^{ij}) \\ &\quad + g(\bar{\nabla}^i\bar{\nabla}^j\alpha - g^{ij}\bar{\nabla}^k\bar{\nabla}_k\alpha). \end{aligned} \tag{8}$$

The slicing density α and the shift β^i are not to be varied. Instead the constraints are imposed on initial data and are preserved dynamically as shown below. This is not an already parametrized theory in the usual sense.

Consider the familiar 3 + 1 identities

$$\dot{g}_{ij} \equiv -2NK_{ij}\,, \tag{9}$$

$$\dot{K}_{ij} \equiv N(\bar{R}_{ij} - R_{ij} + HK_{ij} - K_{ik}K_j{}^k - N^{-1}\bar{\nabla}_i\bar{\nabla}_j N)\,. \tag{10}$$

Also, recall the formula for the derivative of the determinant of the three-metric, $g^{-1}\dot{g} = g^{ij}\dot{g}_{ij} = -2NH$. Now we pass to canonical variables. Using (9) and (10), the time derivative of π^{ij} is computed to be identically

$$\begin{aligned}\dot{\pi}^{ij} &\equiv Ng^{1/2}(\bar{R}g^{ij} - \bar{R}^{ij}) - Ng^{-1/2}(2\pi^{ik}\pi_k{}^j - \pi\pi^{ij}) \\ &\quad + g^{1/2}(\bar{\nabla}^i\bar{\nabla}^j N - g^{ij}\bar{\nabla}^k\bar{\nabla}_k N) + Ng^{1/2}[\mathcal{R}^{ij}], \end{aligned} \tag{11}$$

where $\mathcal{R}_{ij} \equiv R_{ij} - g_{ij}R_k{}^k$.

We see that the equations of motion (7) and (8) derived from the action principle are (9) and (11) when $R^{ij} - g^{ij}R_k{}^k = 0$. Thus, to say that (8) holds is to assert that $R^{ij} = 0$. The equations of motion hold strongly, independent of whether the constraints are satisfied. This is not true in the ADM formulation because of the presence of the Hamiltonian density in their equation of motion for π^{ij}.

This difference can be explained more fully as follows. From the definition of the Einstein tensor in terms of the Ricci tensor, $G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R_\sigma{}^\sigma$, and the observation that $2G_0{}^0 \equiv R_0{}^0 - R_k{}^k$, one derives the identity

$$G_{ij} + g_{ij}G_0{}^0 \equiv R_{ij} - g_{ij}R_k{}^k\,. \tag{12}$$

The vanishing of the right-hand side does not depend on either the Hamiltonian or momentum densities and is equivalent to $R_{ij} = 0$. Clearly, it is also equivalent to $G_{ij} = -g_{ij}G_0{}^0$. Thus, while $R_{\mu\nu} = 0$ and $G_{\mu\nu} = 0$ are equivalent, $R_{ij} = 0$ and $G_{ij} = 0$ are not equivalent as equations of motion—unless the Hamiltonian density $\mathcal{H} = 2g^{1/2}G_0{}^0$ vanishes exactly, that is, unless the Hamiltonian constraint holds. The ADM action principle is equivalent to $G_{ij} = 0$ and so also requires $\mathcal{H} = 0$ to be equivalent to (11). We recall that the use of R_{ij} has always been preferred by the French school, pioneered by Lichnerowicz [23] and Choquet-Bruhat [24].

This raises an important principle: A constrained Hamiltonian theory should be well behaved even when the constraints are violated. As discussed in [25], recent efforts [8–15] to achieve well-posed hyperbolic formulations of general relativity, with only physical characteristics, can be understood in this light as well. From this point of view, the Hamiltonian and momentum densities are definite fixed combinations of the phase space variables, but their values may deviate from zero. The form of the equations of motion should not depend on these values. When the constraints are satisfied (i.e., the densities vanish), one is on the so-called constraint hypersurface, and many unphysical degrees of freedom are frozen because relations among many of the variables $(\mathbf{g}, \pi)$ are fixed. When the constraints are relaxed, the theory explores phase space away from the constraint hypersurface. The objective is to have a theory whose character does not change dramatically when one moves off the constraint hypersurface. Examples for which this is particularly relevant are numerical applications where violation of the constraints is inevitable. It seems that similar properties may be shared by the canonical Ashtekar variables, but there are subtleties beyond the scope of this paper that require closer investigation (cf. [17]). [At this point, we should stress that the $R_{ij} = 0$ equations given by (7) and (8) or (9) and (10) are not in themselves known to be well posed, though they have no unphysical characteristic speeds [7]. They do, however, lead to the well-posed evolution of $\mathcal{H}$ and $\mathcal{H}_i$ as we shall see.]

We now introduce the smeared Hamiltonian

$$\tilde{\mathcal{H}}_\alpha = \int d^3x' \alpha(x',t)\tilde{\mathcal{H}} \,. \qquad (13)$$

The equation of motion for a general functional on phase space $F[\mathbf{g},\pi;x,t)$ is

$$\hat{\partial}_0 F[\mathbf{g},\pi;x,t) = -\{\tilde{\mathcal{H}}_\alpha, F[\mathbf{g},\pi;x,t)\} + \tilde{\partial}_0 F[\mathbf{g},\pi;x,t)\,. \qquad (14)$$

Here, $\hat{\partial}_0$ is a total time derivative while $\tilde{\partial}_0$ is a "partial" derivative of the form $\partial_t - \mathcal{L}_{\boldsymbol{\beta}}$ which only acts on explicit spacetime dependence. The Poisson bracket is given by

$$\{F,G\} = \int d^3x \frac{\delta F}{\delta g_{ij}(x,t)} \frac{\delta G}{\delta \pi^{ij}(x,t)} - \frac{\delta F}{\delta \pi^{ij}(x,t)} \frac{\delta G}{\delta g_{ij}(x,t)}\,.$$

It is evident that the equations of motion (7) and (8) are obtained by applying (14) to the canonical variables $(\mathbf{g},\pi)$. Also, observe that applying (14) to $N = \alpha g^{1/2}$ produces (2).

Time evolution is generated by the Hamiltonian vector field

$$\mathcal{X}_{\tilde{\mathcal{H}}_\alpha} = \int d^3x \bigg\{ \alpha(2\pi_{ij} - \pi g_{ij}) \frac{\delta}{\delta g_{ij}} - [\alpha g(\bar{R}^{ij} - g^{ij}\bar{R}) + \alpha(2\pi^{ik}\pi_k{}^j - \pi\pi^{ij}) - g(\bar{\nabla}^i\bar{\nabla}^j\alpha - g^{ij}\bar{\nabla}^k\bar{\nabla}_k\alpha)] \frac{\delta}{\delta \pi^{ij}} \bigg\}. \qquad (15)$$

Again, this does not depend on the Hamiltonian or momentum densities, so it is a good time evolution operator even away from the constraint hypersurface. (This observation was essentially made by Ashtekar in footnote 17 of [2] but evolution was mistakenly associated with $G_{ij} = 0$).

By the product rule shared by $\hat{\partial}_0$ and the Poisson bracket, one computes the evolution equations for the constraints to be

$$\dot{\tilde{\mathcal{H}}} = -\{\tilde{\mathcal{H}}_\alpha, \tilde{\mathcal{H}}\} = g\alpha g^{ij}\partial_i \mathcal{H}_j + 2gg^{ij}\mathcal{H}_i \bar{\nabla}_j \alpha\,, \qquad (16)$$

$$\dot{\mathcal{H}}_j = -\{\tilde{\mathcal{H}}_\alpha, \mathcal{H}_j\} = \alpha\partial_j\tilde{\mathcal{H}} + 2\tilde{\mathcal{H}}\partial_j\alpha\,, \qquad (17)$$

where $\bar{\nabla}_j\alpha = \partial_j\alpha + \alpha g^{-1/2}\partial_j g^{1/2}$. These equations correspond to (20) and (21) below when $R_{ij} = 0$. Thus, the Poisson brackets of the smeared Hamiltonian with the unsmeared densities are seen to be well-posed evolution equations for the densities.

These equations can be shown to be equivalent to the twice-contracted Bianchi identities $\nabla_\beta G_\alpha{}^\beta \equiv 0$ when $R_{ij} = 0$ as follows. Combine the identity (12) with the twice-contracted Bianchi identities to obtain a transparent form of the Bianchi identities

$$\nabla_\beta G_0{}^\beta \equiv \nabla_0 G_0{}^0 + \nabla_j G_0{}^j \equiv 0\,, \qquad (18)$$

$$\nabla_\beta G_j{}^\beta \equiv \nabla_0 G_j{}^0 - \nabla_j G_0{}^0 + \nabla_i[R_j{}^i - \delta_j{}^i R_k{}^k] \equiv 0\,. \qquad (19)$$

(After the expression in square brackets is replaced using the vanishing of R_{ij}, these become equations of motion rather than identities.) Express the Bianchi identities (18) and (19) in 3 + 1 language, using $C = g^{-1}\tilde{\mathcal{H}} = 2G_0{}^0$ and $C_i = g^{-1/2}\mathcal{H}_i = 2NG_i{}^0$. A calculation yields

$$\dot{C} - N\bar{\nabla}^j C_j \equiv 2(C_j\bar{\nabla}^j N + NHC - NK^{ij}[\mathcal{R}_{ij}])\,, \qquad (20)$$

$$\dot{C}_j - N\bar{\nabla}_j C \equiv 2(C\bar{\nabla}_j N + \tfrac{1}{2}NHC_j - \bar{\nabla}^i(N[\mathcal{R}_{ij}]))\,, \qquad (21)$$

where $\mathcal{R}_{ij} \equiv R_{ij} - g_{ij}R_k{}^k$. This system is clearly symmetric hyperbolic with only the light cone as characteristic, and, changing to $\tilde{\mathcal{H}}$, $\mathcal{H}_i$ gives (16) and (17) when $R_{ij} = 0$. (Without considering our identities or a Hamiltonian framework, Frittelli [26] reached the same conclusion about well posedness of constraint propagation in the "standard" 3 + 1 formulation [19], which uses $R_{ij} = 0$, and its absence in the ADM equations, with $G_{ij} = 0$. Related formulas were also obtained by Choquet-Bruhat and Noutchegueme [27] for the evolution of matter sources ρ^{00}, ρ^{0i}, where $\rho^{\beta\alpha} = T^{\beta\alpha} - \frac{1}{2}g^{\beta\alpha}T_\mu{}^\mu$.)

We now return to the Hamiltonian formulation. The Poisson bracket between two smeared Hamiltonians is

$$\{\tilde{\mathcal{H}}_{\alpha_1}, \tilde{\mathcal{H}}_{\alpha_2}\} = -\int d^3x g(\alpha_1\bar{\nabla}^i\alpha_2 - \alpha_2\bar{\nabla}^i\alpha_1)\mathcal{H}_i = -\int d^3x g g^{ij}(\alpha_1\partial_j\alpha_2 - \alpha_2\partial_j\alpha_1)\mathcal{H}_i\,. \qquad (22)$$

This bracket expresses the consistency of time evolution under different choices of α. The Jacobi identity is

$$\{\tilde{\mathcal{H}}_{\alpha_1}, \{\tilde{\mathcal{H}}_{\alpha_2}, F\}\} - \{\tilde{\mathcal{H}}_{\alpha_2}, \{\tilde{\mathcal{H}}_{\alpha_1}, F\}\} = \{\{\tilde{\mathcal{H}}_{\alpha_1}, \tilde{\mathcal{H}}_{\alpha_2}\}, F\}\,. \qquad (23)$$

Because of the metric dependence in (22), one sees that the difference between evolution with α_2 followed by α_1 and the reverse is a spatial diffeomorphism when $\mathcal{H}_i = 0$ [28] (or when $\delta F/\delta\pi^{ij} = 0$).

These results lead to a new understanding of the Dirac "algebra" of the constraints (cf. [28]). As is well known, the Dirac algebra is not the spacetime diffeomorphism algebra. The root of this is that the action (4) is invariant under transformations generated by the constraints even when they are not satisfied [29]. The equations which hold even when the constraints are not imposed are $R_{ij} = 0$. These equations are preserved by spatial diffeomorphisms and time translations along their flow, yet a general spacetime diffeomorphism applied to $R_{ij} = 0$ mixes in the constraints. A comparison of (16) and (17) and (20) and (21) shows the effect clearly. The Bianchi identities are spacetime diffeomorphism invariant while the constraint evolution equations derived from the action principle are not. The equations (16) and (17) and (20) and (21) differ precisely by terms proportional to R_{ij}.

A second crucial understanding is the way in which the once-smeared form of the Dirac algebra (16) and (17) ensures consistency of the constraints via a well-posed initial value problem. If the constraints vanish initially, then they always vanish in a corresponding physical domain of dependence. This dynamical mechanism for consistency follows from the dual role of $\tilde{\mathcal{H}}$ as part of the generator of time translations and as an initial value constraint.

It is worth reemphasizing the altered role of the Hamiltonian constraint. The Hamiltonian constraint is an initial value constraint from which $g^{1/2}$ is determined as in the solution of the initial value problem [19], which then allows N to be reconstructed from α. By virtue of (16) and (17), once the initial value problem is solved, it remains solved in a spacetime domain dictated by causality. The Hamiltonian constraint does not express the dynamics of the theory; (14) is the dynamical equation. The application of these ideas to canonical quantum gravity will appear elsewhere [30].

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