# Nonlinear Hydrodynamic Stability

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The variational principle of V. I. Arnold [J. Appl. Math. Mech. **29**, 1002 (1965)] is extended to general ideal magnetohydrodynamics (MHD). This is done using the trick of "superdynamics," or the replacement of certain terms in the equations of motion by arbitrary functions. The variational constraint thus introduced leads to a sufficient, and likely necessary, Lyapunov stability criterion. All ideal MHD equilibria with fluid flow, except those with parallel sub-Alfvénic flow, are unstable according to this criterion. The method of superdynamics is extensible to other Hamiltonian systems. [S0031-9007(97)05240-X]

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The standard approach to hydrodynamic stability involves linearization about an equilibrium flow in order to solve for eigenfrequencies [1,2] or to establish a Lyapunov stability criterion for the linearized system [3]. A variety of linear variational principles was developed for both neutral fluids [4] and magnetohydrodynamics (MHD) [5–7], in which the stability criterion is expressed in terms of a positive definite quadratic form. It is well known that the linearized stability does not guarantee the true (Lyapunov) stability, such as in the toy system $du/dt = u^2$, whose equilibrium $u = 0$ is linearly stable.

Nonlinear stability is guaranteed by the presence of an integral of motion, for example, the energy $H$, which assumes a nondegenerate conditional extremum (a minimum or a maximum) subject to the conservation of any other integrals of motion, for example, Casimir invariants [8]. The possibility to write explicitly a full infinite set of integrals is mostly limited to two-dimensional systems. By *explicit* we mean an integral of motion which can be written in terms of the physical fields of velocity, density, etc., in a way which does not require the solution of the equations of the motion. In three dimensions, such integrals are scarce. For example, the Euler equation

$$\partial_t \boldsymbol{\omega} = \nabla \times (\mathbf{v} \times \boldsymbol{\omega}), \quad \boldsymbol{\omega} = \nabla \times \mathbf{v}, \quad \nabla \cdot \mathbf{v} = 0, \tag{1}$$

conserves explicitly only the energy $H$ and the helicity $I$:

$$H = \int \frac{\mathbf{v}^2}{2} d^3\mathbf{x}, \qquad I = \int \mathbf{v} \cdot \boldsymbol{\omega} d^3\mathbf{x}. \tag{2}$$

(Here and below all volume integrals are over the domain occupied by the fluid. An appropriate conservative boundary condition, such as zero normal velocity, is implied.) In addition to the two explicit integrals (2), there is also the infinity of Kelvin invariants,

$$I_\gamma = \oint_\gamma \mathbf{v} \cdot d\ell \equiv \int \boldsymbol{\omega} \cdot d\mathbf{S} = \text{const}, \tag{3}$$

expressing the velocity circulation around (or the vorticity flux through) any closed contour $\gamma(t)$ moving with the fluid velocity $\mathbf{v}$. Integrals (3) are *implicit* in the sense that their definition involves contours $\gamma$ whose motion must be solved from Eq. (1). Although there is no apparent way of incorporating integrals like (3) in a Lyapunov functional, Arnold [9] proposed that the conservation of all vorticity integrals (3) has the geometrical meaning of confining the system to an "isovortical sheet" in the infinite-dimensional phase space. Different sets of initial vorticity integrals specify different sheets such that the whole phase space is "foliated," as if by isosurfaces of an integral of motion (Fig. 1).

The usefulness of the foliation for stability is due to the local explicit parametrization of the isovortical sheets by an incompressible "displacement" $\boldsymbol{\xi}(\mathbf{x})$, such that vorticity fields sharing the sheet with the reference flow $\boldsymbol{\omega}_0(\mathbf{x})$ are written $\boldsymbol{\omega} = \boldsymbol{\omega}_0 + \delta\boldsymbol{\omega}_0 + \frac{1}{2}\delta^2\omega_0 + \cdots$, where

$$\delta\boldsymbol{\omega} = \nabla \times (\boldsymbol{\xi} \times \boldsymbol{\omega}). \tag{4}$$

The linear operator $\delta$ defining the isovortical variation can be derived from the "superdynamics" $\partial_t\boldsymbol{\omega} = \nabla \times (\partial_t\boldsymbol{\xi} \times \boldsymbol{\omega})$, $\nabla \cdot \boldsymbol{\xi} = 0$, in which the vorticity is incompressibly advected in a way similar to the Euler

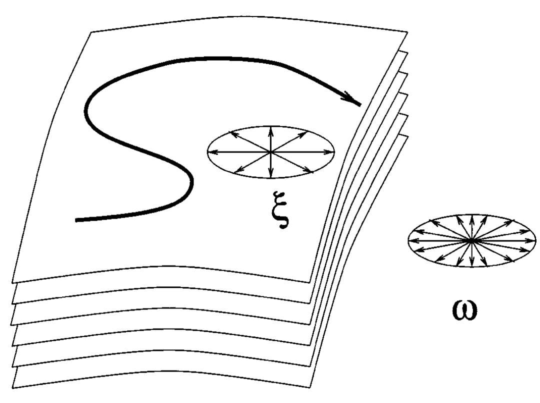


FIG. 1. Schematic of the infinite-dimensional phase space of incompressible fluid flows, $\boldsymbol{\omega}(\mathbf{x})$, which is foliated by isovortical invariant sheets parametrized by the displacement function $\boldsymbol{\xi}(\mathbf{x})$. The dynamics keeps an orbit on a sheet. The arrows are meant as schematic coordinate axes, which could stand for Fourier modes or any other discrete representation of a continuum field.

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equation (1), but by the velocity field $\partial_t \boldsymbol{\xi}$ entirely unrelated to the actual flow $\mathbf{v}$. Since the conservation of the vorticity integrals (3) is independent of the relation between $\mathbf{v}$ and $\boldsymbol{\omega}$, and the real dynamics (1) is a subset of the superdynamics, Eq. (4) follows.

Arnold's variation (4) makes the Hamiltonian $H$ stationary if and only if the flow $\mathbf{v}$ is in equilibrium. Then the second energy variation,

$$\delta^2 H = \int [(\delta \mathbf{v})^2 - \boldsymbol{\xi} \times \boldsymbol{\omega} \cdot \nabla \times (\boldsymbol{\xi} \times \mathbf{v})]\, d^3\mathbf{x}\,, \quad (5)$$

if definite for all incompressible $\boldsymbol{\xi}$, guarantees that the equilibrium is stable [9,10].

Given this long introduction, we briefly report on a generalization of the Arnold method in two important ways. First, our fluid equations (6)–(9) include compressibility, varying entropy, and also magnetic field, but still no dissipation. In such a general formulation, it is difficult to write all integrals generalizing Eq. (3) for arbitrary initial conditions. Therefore, and second, an analog of the isovortical variation is formally derived from the dynamics, without regard to either explicit or implicit integrals of motion. A new outcome of this procedure is an energy principle for ideal MHD stability with fluid flow, a long-standing plasma-physics problem [6,7,11]. Our variational method is similar to that of Friedlander and Vishik [12] who used infinite-dimensional Lie groups and also to that of Morrison [13] who used noncanonical Poisson brackets. In this paper we use the simple physical argument of superdynamics to derive the nonlinear MHD energy principle and to make specific predictions about the stability of MHD equilibria with fluid flow.

We consider the following hydrodynamic equations for an inviscid, ideally conducting fluid:

$$\rho(\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v}) = -\nabla p(\rho, s) + \mathbf{j} \times \mathbf{B} - \rho \nabla \phi\,, \quad (6)$$

$$\partial_t \mathbf{B} = \nabla \times (\mathbf{v} \times \mathbf{B})\,, \quad (7)$$

$$\partial_t s + \mathbf{v} \cdot \nabla s = 0\,, \quad (8)$$

$$\partial_t \rho + \nabla \cdot \rho \mathbf{v} = 0\,. \quad (9)$$

Here $p$ is the fluid pressure, $\rho$ the density, $s$ the entropy, $\phi$ the external gravitational potential, $\mathbf{B}$ the magnetic field, and $\mathbf{j} = \nabla \times \mathbf{B}$ the electric current. The fluid flow conserves the energy

$$H = \int \left( \frac{\rho \mathbf{v}^2}{2} + \rho \epsilon(\rho, s) + \rho \phi + \frac{\mathbf{B}^2}{2} \right) d^3\mathbf{x}\,, \quad (10)$$

where $\epsilon$ is the specific internal energy defined by the standard thermodynamic relation

$$d\epsilon = T\, ds - p d(1/\rho)\,. \quad (11)$$

The varying entropy and the Lorentz force in Eq. (6) break the "frozen-in law" for the vorticity $\boldsymbol{\omega} = \nabla \times \mathbf{v}$, and instead of (1) we now have

$$\partial_t \boldsymbol{\omega} = \nabla \times \left( \mathbf{v} \times \boldsymbol{\omega} + \mathbf{j} \times \frac{\mathbf{B}}{\rho} + \int^{\rho} \frac{\partial_s p\, d\rho}{\rho^2} \nabla s \right). \quad (12)$$

One can introduce superdynamics for Eqs. (6)–(9) in many different ways [14]. Our choice is dictated by the desire to have a zero energy variation for equilibrium flows. After many trials, the following procedure works to our satisfaction: We replace $\mathbf{v} \rightarrow \partial_t \boldsymbol{\xi}$ in Eqs. (7)–(9) and (12). In Eq. (12), we also write $\mathbf{j} \rightarrow \partial_t \boldsymbol{\eta}$ ($\nabla \cdot \boldsymbol{\eta} = 0$) and replace the integral by a scalar $\partial_t \alpha$. The result is *the generalized isovortical variation,*

$$\delta \mathbf{v} = \boldsymbol{\xi} \times \boldsymbol{\omega} + \boldsymbol{\eta} \times \frac{\mathbf{B}}{\rho} + \alpha \nabla s + \nabla \beta\,, \qquad \boldsymbol{\eta} = \nabla \times \boldsymbol{\zeta}\,, \quad (13)$$

$$\delta \mathbf{B} = \nabla \times (\boldsymbol{\xi} + \mathbf{B}), \qquad \delta s = -\boldsymbol{\xi} \cdot \nabla s\,, \qquad \delta \rho = -\nabla \cdot \rho \boldsymbol{\xi}\,, \quad (14)$$

which depends on two arbitrary vectors $\boldsymbol{\xi}$ and $\boldsymbol{\zeta}$ and two arbitrary scalars $\alpha$ and $\beta$. The boundary condition for $\boldsymbol{\xi}$ is the same as for $\mathbf{v}$ (zero normal component); no boundary conditions are imposed on $\boldsymbol{\zeta}$, $\alpha$, and $\beta$.

A straightforward check shows that the variation (13) and (14) conserves an infinite set of local integrals, including (a) the entropy of each fluid element, (b) the magnetic flux through each material (i.e., moving with the fluid) contour, (c) the total cross-helicity $\mathbf{v} \cdot \mathbf{B}$ within each magnetic flux tube, if, initially, such tubes exist and the entropy $s$ is constant on the magnetic flux surfaces. [According to (a) and (b), the isosurfaces of $s$ and magnetic flux surfaces are material.] These are all (except the energy) known integrals conserved by ideal MHD and locally expressible through the physical fields $(\mathbf{v}, \mathbf{B}, s, \rho)$. Whether or not there are other nontrivial local integrals conserved by the variation $\delta$ we do not know; nevertheless, the way $\delta$ is introduced clearly implies that the phase-space sheets parametrized by $(\boldsymbol{\xi}, \boldsymbol{\zeta}, \alpha, \beta)$ are invariant sheets, which can be interpreted as isosurfaces of some combination of integrals of motion and thus used for stability analysis. This combination of the integrals is very likely *complete,* except for the energy, which will be varied individually subject to the constraints built in the variation $\delta$.

The number of arbitrary functions in the variation (13) and (14) by no accident equals the number of dynamical equations (6)–(9). Upon varying the energy (10) and using Eq. (11), a few integrations by parts yield

$$\delta H = \int [\boldsymbol{\xi} \cdot (\rho \mathbf{v} \cdot \nabla \mathbf{v} + \nabla p - \mathbf{j} \times \mathbf{B} + \rho \nabla \phi) - \boldsymbol{\zeta} \cdot \nabla \times (\mathbf{v} \times \mathbf{B}) + \alpha \rho \mathbf{v} \cdot \nabla s - \beta \nabla \cdot \rho \mathbf{v}]\, d^3\mathbf{x}\,. \quad (15)$$

By design, the condition that $\delta H = 0$ for all $(\boldsymbol{\xi}, \boldsymbol{\zeta}, \alpha, \beta)$ is equivalent to an equilibrium solution of Eqs. (6)–(9).

The second variation of the velocity [15],

$$\delta^2 \mathbf{v} = \boldsymbol{\xi} \times \delta\boldsymbol{\omega} + \boldsymbol{\eta} \times \delta \frac{\mathbf{B}}{\rho} + \alpha \nabla \delta s + \nabla \beta, \quad (16)$$

and similar expressions for $\delta^2(\mathbf{B}, s, \rho)$ are now used to calculate the second energy variation:

$$\delta^2 H = \int \left[ \delta^2\left(\rho\epsilon + \frac{\mathbf{B}^2}{2}\right) + \left(\phi + \frac{\mathbf{v}^2}{2}\right)\delta^2\rho + \rho(\delta\mathbf{v})^2 + \rho\mathbf{v}\cdot\delta^2\mathbf{v} + 2\delta\rho\mathbf{v}\cdot\delta\mathbf{v}\right] d^3\mathbf{x}. \quad (17)$$

Equations (13) and (14) define $\delta^2 H$ as a functional of $(\boldsymbol{\xi}, \boldsymbol{\zeta}, \alpha, \beta)$. Two comments regarding the form of $\delta^2 H$ are in order.

First, the suspicious linear term $\nabla\beta$ in the second velocity variation (16) is dotted with an incompressible $\rho\mathbf{v}$ in Eq. (17) and thus vanishes upon integration by parts. So, as it should be, $\delta^2 H$ is a *quadratic* functional of the independent variables $(\boldsymbol{\xi}, \boldsymbol{\zeta}, \alpha, \beta)$.

Second, the integrand of (17) can be written as a quadratic polynomial of $\alpha$ with the coefficient $\rho(\nabla s)^2$ in front of $\alpha^2$. Therefore, the definite sign of $\delta^2 H$ can be only positive, and, for this, it is necessary and sufficient that the $\alpha$-minimized quadratic form be positive:

$$W \equiv \min_{\alpha} \delta^2 H = W_{\mathrm{static}}(\boldsymbol{\xi}) + W_{\mathrm{flow}}(\boldsymbol{\xi}, \boldsymbol{\zeta}, \beta) > 0. \quad (18)$$

Here the following notation is introduced:

$$W_{\mathrm{static}} \equiv \int \delta^2(\rho\phi + \rho\epsilon + \mathbf{B}^2/2)\, d^3\mathbf{x} = \int [\phi\delta^2\rho + \nabla\cdot\boldsymbol{\xi}(\rho\partial_\rho p \nabla\cdot\boldsymbol{\xi} + \boldsymbol{\xi}\cdot\nabla p) + \delta\mathbf{B}\cdot(\delta\mathbf{B} - \boldsymbol{\xi}\times\mathbf{j})]\, d^3\mathbf{x}, \quad (19)$$

where $\delta^2\rho = -\nabla\cdot\delta\rho\boldsymbol{\xi}$. The flow part of $W$ is

$$W_{\mathrm{flow}} = \int \left[ \frac{\mathbf{v}^2}{2}\delta^2\rho - \rho(\mathbf{n}\cdot\delta''\mathbf{v})^2 + \delta'\mathbf{v}\cdot(\rho\delta''\mathbf{v} + \mathbf{v}\delta\rho) + \rho\mathbf{v}\cdot\boldsymbol{\eta}\times\delta\frac{\mathbf{B}}{\rho}\right] d^3\mathbf{x}, \quad (20)$$

where $\mathbf{n} = \nabla s/|\nabla s|$ is the normal to the magnetic/fluid flux surfaces, $\delta(\mathbf{B}/\rho) = (\mathbf{B}/\rho)\cdot\nabla\boldsymbol{\xi} - \boldsymbol{\xi}\cdot\nabla(\mathbf{B}/\rho)$, and

$$\delta'\mathbf{v} \equiv \boldsymbol{\xi}\times\boldsymbol{\omega} + \boldsymbol{\eta}\times\mathbf{B}/\rho + \nabla\beta,$$

$$\delta''\mathbf{v} \equiv \delta'\mathbf{v} + \boldsymbol{\xi}\cdot\nabla\mathbf{v} - \mathbf{v}\cdot\nabla\boldsymbol{\xi}.$$

No further "simple" minimization of Eq. (20) is possible in general. (In the case of a parallel flow, $\mathbf{v}\|\mathbf{B}$, $W$ can be also minimized with respect to $\boldsymbol{\eta} = \nabla\times\boldsymbol{\zeta}$.)

Since all known explicit and implicit integrals of motion have been accounted for (the possibly conserved linear and angular momenta amount to choosing an appropriate frame of reference), *we propose that the sufficient stability criterion $W(\boldsymbol{\xi}, \boldsymbol{\zeta}, \beta) > 0$ is also necessary for the true nonlinear stability of an ideal MHD equilibrium.* This conjecture is supported by the static limit of zero flow, $\mathbf{v} = 0$, in which our energy principle reduces to Eq. (19), or the standard MHD energy principle [5], whose violation means a linear instability. As a byproduct, we thus find that the linear stability criterion of Bernstein *et al.* [5] for static equilibria is also a nonlinear stability criterion. In a general situation with fluid flow, an indefinite $W$ may not result in an exponential instability, but rather lead to a slower, algebraic perturbation growth and subsequent turbulence. This scenario will be described elsewhere.

The other two limiting cases we would like to mention are (a) the hydrostatic equilibrium with $\phi = gz$ and $\mathbf{v} = \mathbf{B} = 0$ and (b) the incompressible neutral fluid with $\rho = s = \mathrm{const}$ and $\mathbf{B} = 0$. For the former case, the condition $W > 0$ yields the well known convective stability criterion [2]: $ds/dz > 0$ and $d\rho/dz < 0$. In the Euler limit, the incompressibility is introduced by letting the sound speed $c^2 = \partial_\rho p$ to infinity. The minimum of the pressure terms in Eq. (19) then implies $\nabla\cdot\boldsymbol{\xi} \to 0$ for the "most dangerous" perturbations, and further minimization of (17) with respect to $\beta$ results in $\nabla\cdot\delta\mathbf{v} = 0$ and the restricted Arnold criterion that Eq. (5) be *positive* definite. We note that in the limit of zero magnetic field the energy principle of Ref. [16] does not reduce to a correct stability criterion for a neutral fluid.

An important conclusion can be drawn from Eqs. (19) and (20), if the necessary status of our energy principle is adopted. It is that *almost all dynamic 3D MHD equilibria are unstable.* The term "almost all" refers to equilibria with the flow $\mathbf{v}$ nonparallel to the magnetic field $\mathbf{B}$ (up to a rigid-body rotation for axisymmetric equilibria). Indeed, for a very short-scale field $\boldsymbol{\xi}$, the perturbation energy to leading order,

$$W = \int \{(\mathbf{B}\cdot\nabla\boldsymbol{\xi})^2 - \rho[\mathbf{n}\cdot(\mathbf{v}\cdot\nabla\boldsymbol{\xi})]^2 + \cdots\}\, d^3\mathbf{x}, \quad (21)$$

can be made either positive or negative by a suitable choice of $\boldsymbol{\xi}$, unless $\mathbf{v}\|\mathbf{B}$ and $\rho v^2 < B^2$. One of the implications of this conclusion is that it is not necessary to invoke the magnetic field perpendicular to an accretion disk in order to have an MHD instability of the laminar accretion [17]; the more generic small azimuthal (parallel) magnetic field will drive the super-Alfvénic fluid flow nonlinearly unstable.

For fusion applications, the conclusion about the MHD instability of rotating plasmas should not be discouraging, because tokamak plasmas are almost collisionless and not accurately described by the MHD equations. For static equilibria, the difference between the collisionless [18] and the MHD [5] energy principles is of order the pressure terms in Eq. (19). For a typical tokamak, the kinetic energy of plasma rotation is of order the thermal energy, or in the same order where the MHD approximation errs.

The superdynamics-based nonlinear stability method can be extended to other Hamiltonian partial differential equation systems. Consider, for example, the collisionless Vlasov equation for the distribution function $f(\mathbf{z}, t)$, with $\mathbf{z} = (\mathbf{x}, \mathbf{p}, \alpha)$ being the collection of the canonical coordinates and momenta and the plasma species label $\alpha$:

$$\partial_t f = [h, f] \equiv \partial_{\mathbf{x}} h \cdot \partial_{\mathbf{p}} f - \partial_{\mathbf{p}} h \cdot \partial_{\mathbf{x}} f \,. \qquad (22)$$

The single-particle (mass $m$, charge $e$) Hamiltonian

$$h(\mathbf{z}, t) = \frac{1}{2m}\left[\mathbf{p} - \frac{e}{c}\mathbf{A}(\mathbf{x}, t)\right]^2 + e\phi(\mathbf{x}, t) \qquad (23)$$

is coupled to $f$ via the Maxwell equations for the electromagnetic potentials $\phi$ and $\mathbf{A}$. The superdynamics for the Vlasov equation is introduced by replacing the self-consistent $h$ in Eq. (22) by an arbitrary scalar $\partial_t \xi(\mathbf{z}, t)$. The resulting variation of the distribution function, $\delta f = [\xi, f]$, automatically conserves an infinity of Casimir invariants $\int C(f, \alpha)\, d^6\mathbf{z}$. (The $\mathbf{z}$ integration includes the summation over $\alpha$.) Then the first variation of the total plasma energy is zero if and only if there is an equilibrium, $\delta H = \int \xi [h, f]\, d^6\mathbf{z}$, and the second variation gives an energy principle in the form of a quadratic functional of $\xi(\mathbf{z})$. The detailed investigation of this and other kinetic energy principles will be presented elsewhere.

The exact mathematical meaning of the generalized isovortical variation (13) and (14) and the status of the resulting stability criterion (18) remain unclear to this author. For example, no *a priori* estimates exist for three-dimensional hydrodynamic perturbations, unlike those in two dimensions, where all Casimir integrals are explicit [8,10,19]. On the "physical level," the sufficient stability criterion (18) looks rigorous. To prove the less rigorous conjecture, namely, that the condition (18) is also necessary for nonlinear stability, will require to develop an analytical theory of nonlinear (that is, nonexponential) ideal hydrodynamic instabilities. At this time, very little is known about such instabilities for a generic hydrodynamic equilibrium, although some interesting examples exist [20]. For a nonlinear instability to occur, it appears important that the corresponding linear continuum spectrum have both positive- and negative-energy modes [13]. The nonlinear interaction of the continuum modes will likely lead to an algebraic growth, as suggested by the universal algebraic damping of such modes in the stable case [21].

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[14] This ambiguity is already present in the Euler equation. For example, one can get a different superdynamics by replacing $\boldsymbol{\omega}$, and not $\mathbf{v}$, by $\partial_t \boldsymbol{\xi}$ in Eq. (1). As a result, the variation $\delta_1 \boldsymbol{\omega} = \nabla \times (\mathbf{v} \times \boldsymbol{\xi})$ defines a new foliation of the phase space by different invariant sheets. Interestingly, the new variation $\delta_1$ keeps energy $H$ constant to all orders, but makes the helicity $I$ stationary only for equilibrium flows and thus allows one to study their stability by inspecting the second helicity variation $\delta_1^2 I$. This observation was made by A. V. Gruzinov.

[15] It should be noted that the operator $\delta$ acting on the fields $(\mathbf{v}, \mathbf{B}, s, \rho)$, Eqs. (13) and (14), is not linear, whereas the linearity of $\delta$ is necessary to derive the finite variation operator, $\delta + \delta^2/2 + \cdots$, defining the generalized isovortical sheet. This problem can be fixed by using a different set of fields, $(\mathbf{v}, \mathbf{b}, s, \rho)$, where $\mathbf{b} = \mathbf{B}/\rho$ is evolved according to $\partial_t \mathbf{b} = \mathbf{b} \cdot \nabla \mathbf{v} - \mathbf{v} \cdot \nabla \mathbf{b}$ and varied as $\delta \mathbf{b} = \mathbf{b} \cdot \nabla \boldsymbol{\xi} - \boldsymbol{\xi} \cdot \nabla \mathbf{b}$. In the new variables, the operator $\delta$ is linear, and the second energy variation is the same as in the text.

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