Analytic Solution of the Pion-Laser Model

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Brooding over bosons, wave packets, and Bose-Einstein correlations, we find that a generalization of the pion-laser model for the case of overlapping wave packets is *analytically solvable* with complete *n*-particle symmetrization. Explicit *multiplicity and momentum dependence* of exclusive correlations and spectra is found. The effective source radii are *reduced for low values* and *enlarged for high values of the mean momentum* in the rare gas limiting case. [S0031-9007(97)05015-1]

The study of the statistical properties of quantum systems has a long history with important recent developments. In high energy physics, quantum statistical correlations are studied in order to infer the space-time dimensions of the elementary particle reactions. In high energy heavy ion collisions hundreds of bosons are created in the present CERN SPS reactions when Pb + Pb reactions are measured at $160A$ GeV laboratory bombarding energy. At the RHIC accelerator, to be completed by 1999, thousands of pions could be produced in a unit rapidity interval [1,2]. If the number of pions in a unit value of phase space is large enough, these bosons may condense into the same quantum state and a pion laser could be created [3]. Similar to this process, when a large number of bosonic atoms are collected in a magnetic trap and cooled down to increase their density in phase space, the bosonic nature of the atoms reveals itself in the formation of a Bose-Einstein condensate [4], a macroscopic quantum state. Such a condensation mechanism may provide the key to the formation of atomic lasers in condensed matter physics and to the formation of pion lasers in high energy particle and heavy ion physics, reviewed recently in Refs. [5–7].

The density matrix of a generic quantum mechanical system is

$$\hat{\rho} = \sum_{n=0}^{\infty} p_n \hat{\rho}_n, \tag{1}$$

where the index n characterizes subsystems with particle number fixed to n, and the multiplicity distribution is prescribed by the set of $\{p_n\}_{n=0}^{\infty}$, normalized as $\sum_{n=0}^{\infty} p_n = 1$. The density matrices are normalized as $\mathrm{Tr}\,\hat{\rho} = 1$ and $\mathrm{Tr}\,\hat{\rho}_n = 1$, where

$$\hat{\rho}_n = \int \prod_{i=1}^{n} d\alpha_i\, \rho_n(\alpha_1,\ldots,\alpha_n)\, |\alpha_1,\ldots,\alpha_n\rangle\langle\alpha_1,\ldots,\alpha_n| \tag{2}$$

and the states $|\alpha_1,\ldots,\alpha_n\rangle$ denote properly normalized n-particle wave-packet boson states.

A wave-packet creation operator is

$$\alpha_i^{\dagger} = \int \frac{d^3\mathbf{p}}{(\pi\sigma^2)^{3/4}}\, e^{-[(\mathbf{p}-\pi_i)^2/2\sigma_i^2] - i\xi_i(\mathbf{p}-\pi_i) + i\omega(\mathbf{p})(t-t_i)}\, \hat{a}_{\mathbf{p}}^{\dagger}, \tag{3}$$

where $\alpha_i = (\xi_i, \pi_i, \sigma_i, t_i)$ refers to the parameters of the wave packet i: the center in space, the center in momentum space, the width in momentum space, and the production time, respectively. For simplicity we assume that all the wave packets are emitted at the same instant and with the same width, $\alpha_i = (\pi_i, \xi_i, \sigma, t_0)$.

The n boson states, normalized to unity, are given as

$$|\alpha_1,\ldots,\alpha_n\rangle = \left(\sum_{\sigma^{(n)}} \prod_{i=1}^{n} \langle\alpha_i|\alpha_{\sigma_i}\rangle\right)^{1/2} \alpha_n^{\dagger}\ldots\alpha_1^{\dagger}|0\rangle. \tag{4}$$

Here $\sigma^{(n)}$ denotes the set of all the permutations of the indexes $\{1, 2, \ldots, n\}$, and the subscript σ_i denotes the index that replaces the index i in a given permutation from $\sigma^{(n)}$.

Solution for a new type of density matrix.—There is one special density matrix, for which one can overcome the difficulty, related to the nonvanishing overlap of many hundreds of wave packets, even in an explicit analytical manner. Namely, if one assumes that we have a system in which the emission probability of a boson is increased if there is another emission in the vicinity,

$$\rho_n(\alpha_1,\ldots,\alpha_n) = \frac{1}{\mathcal{N}(n)} \prod_{i=1}^{n} \rho_1(\alpha_i) \left(\sum_{\sigma^{(n)}} \prod_{k=1}^{n} \langle\alpha_k|\alpha_{\sigma_k}\rangle\right). \tag{5}$$

The coefficient of proportionality, $\mathcal{N}(n)$, can be determined from the normalization condition. The density matrix of Eq. (5) describes a quantum-mechanical wave-packet system with induced emission, the amount of the induced emission is controlled by the overlap of the n wave packets [8], yielding a weight in the range of $[1, n!]$. Although it is very difficult numerically to operate with such a wildly fluctuating weight, we were able to reduce the problem [8] to an already discovered "ring" algebra of permanents for plane-wave outgoing states [3].

For the sake of simplicity we consider nonrelativistic particles, $\omega(\mathbf{k}) = \mathbf{k}^2/(2m)$. In order to define an analytically solvable model, we also assume a nonrelativistic, nonexpanding static source at rest in the frame where the calculations are performed,

$$\rho_1(\alpha) = \rho_x(\xi)\rho_p(\pi)\delta(t - t_0),$$

$$\rho_x(\xi) = \frac{1}{(2\pi R^2)^{3/2}} \exp(-\xi^2/2R^2), \qquad (6)$$

$$\rho_p(\pi) = \frac{1}{(2\pi mT)^{3/2}} \exp(-\pi^2/2mT),$$

and a Poisson multiplicity distribution $p_n^{(0)}$ for the case when the Bose-Einstein effects are negligible,

$$p_n^{(0)} = \frac{n_0^n}{n!} \exp(-n_0). \qquad (7)$$

This corresponds to the very rare gas limit, and completes the specification of the model. The plane-wave model, to which the multiparticle wave-packet model was reduced in Ref. [8], can be further simplified [8] to a set of recurrence relations with the help of the so-called "ring algebra" discovered first by Pratt in Ref. [3]. The probability of finding events with multiplicity n, as well as the single-particle and the two-particle momentum distribution in such events, is given as

$$p_n = \omega_n \left(\sum_{k=0}^{\infty} \omega_k \right)^{-1}, \qquad (8)$$

$$N_1^{(n)}(\mathbf{k}_1) = \sum_{i=1}^{n} \frac{\omega_{n-i}}{\omega_n} G_i(1,1), \qquad (9)$$

$$N_2^{(n)}(\mathbf{k}_1, \mathbf{k}_2) = \sum_{l=2}^{n} \sum_{m=1}^{l-1} \frac{\omega_{n-l}}{\omega_n} [G_m(1,1)G_{l-m}(2,2) + G_m(1,2)G_{l-m}(2,1)], \qquad (10)$$

where $\omega_n = p_n/p_0$ and

$$G_n(i, j) = n_0^n h_n \exp[-a_n(\mathbf{k}_i^2 + \mathbf{k}_j^2) + g_n \mathbf{k}_i \mathbf{k}_j]. \qquad (11)$$

Averaging over the multiplicity distribution p_n yields the inclusive spectra as

$$G(1, 2) = \sum_{n=1}^{\infty} G_n(1, 2), \qquad (12)$$

$$N_1(\mathbf{k}_1) = \sum_{n=1}^{\infty} p_n N_1^{(n)}(\mathbf{k}_1) = G(1, 1), \qquad (13)$$

$$N_2(\mathbf{k}_1, \mathbf{k}_2) = G(1, 1)G(2, 2) + G(1, 2)G(2, 1). \qquad (14)$$

An auxiliary quantity is introduced as

$$C_n = \frac{1}{n} \int d^3\mathbf{k}_1 \, G_n(1, 1) = h_n \frac{n_0^n}{n} \left(\frac{\pi}{2a_n - g_n} \right)^{3/2}. \qquad (15)$$

With the help of the notation

$$\sigma_T^2 = \sigma^2 + 2mT, \qquad R_e^2 = R^2 + \frac{mT}{\sigma^2 \sigma_T^2}, \qquad (16)$$

the recurrence relations that correspond to the solution of the ring algebra [3,9] are obtained [8] for the case of the multiparticle wave-packet model. These correspond to the

pion-laser model of Pratt when a replacement $R \rightarrow R_e$ and $T \rightarrow T_e = \sigma_T^2/(2m)$ is performed.

Let us introduce the following auxiliary quantities:

$$\gamma_{\pm} = \frac{1}{2}(1 + x \pm \sqrt{1 + 2x}), \qquad x = R_e^2 \sigma_T^2. \quad (17)$$

The *general analytical solution* of the model is given through the generating function of the multiplicity distribution p_n

$$G(z) = \sum_{n=0}^{\infty} p_n z^n = \exp\left(\sum_{n=1}^{\infty} C_n(z^n - 1) \right), \qquad (18)$$

where C_n is

$$C_n = \frac{n_0^n}{n} [\gamma_+^{n/2} - \gamma_-^{n/2}]^{-3}, \qquad (19)$$

together with the *general analytic solution* for the functions $G_n(1, 2)$,

$$G_n(1, 2) = j_n \exp\left\{ -\frac{b_n}{2} [(\gamma_+^{n/2}\mathbf{k}_1 - \gamma_-^{n/2}\mathbf{k}_2)^2 + (\gamma_+^{n/2}\mathbf{k}_2 - \gamma_-^{n/2}\mathbf{k}_1)^2] \right\}, \qquad (20)$$

$$j_n = n_0^n \left(\frac{b_n}{\pi} \right)^{3/2}, \qquad b_n = \frac{1}{\sigma_T^2} \frac{\gamma_+ - \gamma_-}{\gamma_+^n - \gamma_-^n}. \qquad (21)$$

The detailed proof that the analytic solution to the multiparticle wave-packet model is indeed given by the above equations is described in Ref. [8].

The representation of Eq. (18) indicates that the quantities C_n are the so-called combinants [10–12] of the probability distribution of p_n and in our case their explicit form is known for any set of model parameters, as given by Eqs. (19), (17), and (16). The resulting multiplicity generating function does not correspond to discrete probability generating functions in Ref. [13], so we have found a new type of probability generating functions.

The mean multiplicity is $\langle n \rangle = \sum_{n=1}^{\infty} n p_n = \sum_{i=1}^{\infty} i C_i$, Ref. [8]. The large n behavior of nC_n depends on the ratio of $N_0/\gamma_+^{3/2}$, since for large values of n we always have $(\gamma_-/\gamma_+)^{n/2} \ll 1$. The critical value of n_0 is

$$n_c = \gamma_+^{3/2} = (1 + x + \sqrt{1 + 2x})^{3/2}/2^{3/2}. \qquad (22)$$

If $n_0 < n$, one finds $\lim_{n\to\infty} nC_n = 0$ and $\langle n \rangle < \infty$; if $n_0 > n_c$ one obtains $\lim_{n\to\infty} nC_n = \infty$ and $\langle n \rangle = \infty$; and finally, if $n_0 = n_c$ one finds $\lim_{n\to\infty} nC_n = 1$ and $\langle n \rangle = \infty$. The divergence of the mean multiplicity $\langle n \rangle$ is related to condensation of the wave packets to the wave-packet state with zero mean momentum, i.e., $\pi = 0$ if $n_0 \geq n_c$, Ref. [8]. The multiplicity distribution of Eq. (18) is studied at greater length in Ref. [8].

Dense gas limiting case.—This wave-packet model exhibits a lasing behavior in the dense Bose-gas limit, corresponding to an optically coherent behavior, characterized by a vanishing enhancement of the two-particle intensity correlations at low momentum, $C(\mathbf{k}_1, \mathbf{k}_2) = 1$, a case which is described in greater detail in Ref. [8].

Rare gas limiting case.—Large source sizes or large effective temperatures correspond to the $x \gg 1$ limiting case, where the general analytical solution of the model, presented above, becomes particularly simple and the exclusive and inclusive spectra and correlation functions can be obtained analytically to leading order in $1/x \ll 1$. From Eq. (11) one obtains that

$$G_n(1,2) = j_n \exp\left(-\frac{n}{2\sigma_T^2}(\mathbf{k}_1^2 + \mathbf{k}_2^2) - \frac{R_e^2}{2n}\Delta\mathbf{k}^2\right), \quad (23)$$

$$j_n = \frac{n^{5/2}C_n}{(\pi\sigma_T^2)^{3/2}}, \qquad C_n = \frac{n_0^n}{n^4}\left(\frac{2}{x}\right)^{(3/2)(n-1)}, \quad (24)$$

where $\Delta\mathbf{k} = \mathbf{k}_1 - \mathbf{k}_2$. We can see from Eq. (23) that the higher order corrections will contribute to the observables with reduced effective temperatures and reduced effective radii. Equation (24) indicates that the leading order combinant in the $x \gg 1$ limiting case is C_1 with the first subleading correction given by C_2. Thus, the probability distribution can be considered in the rare gas limiting case as a Poisson distribution of particle singlets with a subleading correction that yields a convolution of Poisson-distributed doublets. The *very* rare gas limiting case corresponds to keeping only the leading $n = 1$ order terms in the above equations. The multiplicity distribution is a Poisson distribution with $\langle n \rangle = n_0$ and no influence from stimulated emission. The momentum distribution is a static Boltzmann distribution, and the exclusive and inclusive momentum distributions coincide [8]. The leading order two-particle Bose-Einstein correlation function is a static Gaussian correlation function with a constant intercept parameter of $\lambda = 1$ and with a momentum-independent radius parameter of $R_* = R_e$ [8].

The probability generating function yields the following leading order multiplicity distribution:

$$p_n = \frac{n_0^n}{n!}\exp(-n_0)\left(1 + \frac{n(n-1) - n_0^2}{2(2x)^{3/2}}\right). \quad (25)$$

The mean multiplicity, the factorial cumulant moments of the multiplicity distribution, and the inclusive and exclusive momentum distributions were evaluated by keeping only the leading order terms in $1/x$ in Ref. [8]. The two-particle exclusive correlation functions can also be evaluated by applying a Gaussian approximation to the leading order corrections in the $x \gg 1$ limiting case,

$$C_2^{(n)}(\mathbf{k}_1, \mathbf{k}_2) = \frac{n^2}{n(n-1)} \frac{N_2^{(n)}(\mathbf{k}_1, \mathbf{k}_2)}{N_1^{(n)}(\mathbf{k}_1)N_1^{(n)}(\mathbf{k}_2)}$$
$$= 1 + \lambda_{\mathbf{K}}\exp(-R_{\mathbf{K},s}^2\Delta\mathbf{k}_s^2 - R_{\mathbf{K},o}^2\Delta\mathbf{k}_o^2), \quad (26)$$

where $\mathbf{K} = 0.5(\mathbf{k}_1 + \mathbf{k}_2)$, the sidewards and outwards directions are introduced for spherical symmetric sources as $\Delta\mathbf{k}_s = \Delta\mathbf{k} - \mathbf{K}(\Delta\mathbf{k} \cdot \mathbf{K})/(\mathbf{K} \cdot \mathbf{K})$ and

$\Delta\mathbf{k}_o = \mathbf{K}(\Delta\mathbf{k} \cdot \mathbf{K})/(\mathbf{K} \cdot \mathbf{K})$, similar to Refs. [14]. The momentum-dependent intercept and radius parameters are

$$\lambda_{\mathbf{K}} = 1 + \frac{2 - 2^{7/2}e^{-\mathbf{K}^2/\sigma_T^2}}{(2x)^{3/2}}, \quad (27)$$

$$R_{\mathbf{K},s}^2 = R_e^2$$
$$+ \frac{R_e^2 - 2^{1/2}e^{-\mathbf{K}^2/\sigma_T^2}[(n+2)R_e^2 + (2/\sigma_T^2)]}{(2x)^{3/2}}, \quad (28)$$

$$R_{\mathbf{K},o}^2 = R_{\mathbf{K},s}^2 + \frac{n}{x^{3/2}}\frac{\mathbf{K}^2}{\sigma_T^4}e^{-\mathbf{K}^2/\sigma_T^2}. \quad (29)$$

Thus the symmetrization results in a momentum-dependent intercept parameter $\lambda_{\mathbf{K}}$ that starts from a $\lambda_{\mathbf{K}=0} < 1$ value at low momentum and *increases* with increasing momentum. Already in the first paper about the pion laser model, Ref. [3], a reduction of the exact intercept parameter was observed and interpreted as the onset of a coherent behavior in the low momentum modes. First, a partially coherent system is created, characterized by $\lambda_{\mathbf{K}} < 1$, and if the density of pions is further increased, one finds a fully developed pion laser with $\lambda_{\mathbf{K}} = 0$; see Ref. [8] for analytic considerations. Although a decrease of the intercept parameter was reported in earlier numerical investigations [15], and a momentum-dependent decrease of the radius parameters was noted for the plane-wave version of the present model in Ref. [16], these investigations were restricted to a few points in the parameter space only and no analytical formulation of these numerical results was given. Because of this, many interesting effects like the increase of the radii and the intercept parameter at large $\mathbf{K}$ were not even expected from the earlier numerical studies. From our analytical solution, however, it follows that the radius parameters not only decrease at low mean momentum, but they also increase at high mean momentum, as compared to R_e. This decrease of the effective source radii, given by Eqs. (28) and (29), is more pronounced for higher values of the fixed multiplicity n, in contrast to the momentum dependence of $\lambda_{\mathbf{K}}$ that is independent of n.

Last, but not least, a specific term appears in the two-particle exclusive correlation function that contributes only to the *out* direction, which, in case of spherically symmetric sources, may be identified with the direction of the mean momentum [14]. This directional dependence is related only to the direction of the relative momentum as compared to the direction of the mean momentum, and does not violate the assumed spherical symmetry of the boson source. This effect vanishes both at very low or at very high values of the mean momentum $\mathbf{K}$, according to Eq. (29). The top, middle, and bottom panels of Fig. 1 indicate the momentum-dependent $\lambda_{\mathbf{K}}$ intercept parameter, and the $R_{\mathbf{K},s}$ and $R_{\mathbf{K},o}$ radius parameters for a fireball with $R = 11$ fm, $T = 120$ MeV. The pions are assumed to be described by wave packets with spatial widths of $\sigma_x = 2$ fm, and events with fixed
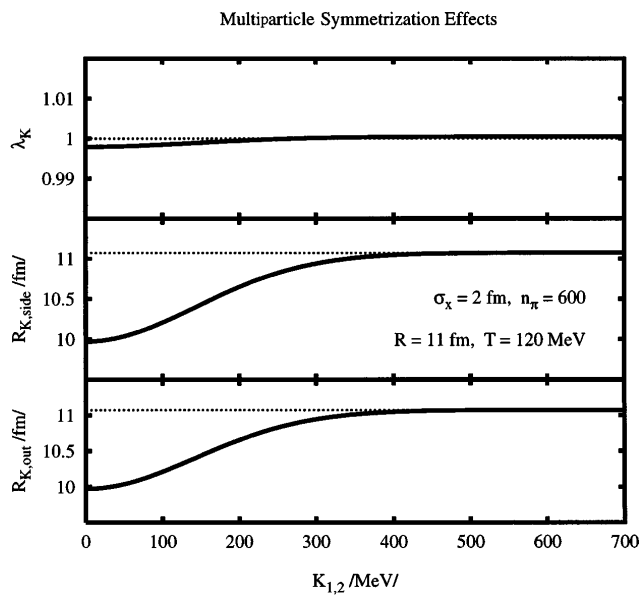
FIG. 1.   Multiparticle symmetrization results at low **K** in a momentum-dependent reduction of the intercept parameter $\lambda_\mathbf{K}$, the sidewards and the outwards radius parameters, $R_{\mathbf{K},s}$ and $R_{\mathbf{K},o}$ from their static values of 1 and R_e, respectively. The enhancement of these parameters at high momentum is hardly noticeable for large and hot systems.

is found to be the most apparent effect. The directional dependence of the radius parameters and the enhancement of the radii at high momentum is characteristic for a small, cold pion gas with only a handful of particles in it. These results can be understood qualitatively by an enhancement of the wave packets in the low momentum modes, due to multiparticle Bose-Einstein symmetrization effects, as the system starts to approach the formation of a laser, characterized by the appearance of partial optical coherence in the low momentum modes.

Our results explicitly depend on the multiplicity, providing theoretical insights to event-by-event analysis of heavy ion data. However, a direct comparison of the model with experimental data is limited to the nonrelativistic kinematical domain, for example, soft pions at midrapidity, for systems where final state interactions and spin and isospin effects are negligible.

identical pion multiplicity of $n_x = 600$ are selected for the evaluation of the correlation function. For this set of parameters, the enhancement of $\lambda_\mathbf{K}$, $R_{\mathbf{K},\text{side}}$ and $R_{\mathbf{K},\text{out}}$ is hardly noticeable at high momentum, but their small decrease at low momentum is clear. One may consider a small cold pionic system with a few large wave packets only, by inserting $R = 4$ fm, $T = 10$ MeV, $n_\pi = 3$, and $\sigma_x = 5$ fm to Eqs. (27)–(29). This source could correspond to heavy ion collisions in the $30A$ MeV energy domain [14] characterized by an effective radius $R_e = 4.5$ fm and effective temperature of $T_e = 15.6$ MeV [14]. In this case, the directional dependence of the radii and the enhancement of the radius parameters at high momentum as compared to R_e becomes significant not only analytically but numerically as well.

Highlights.—In this Letter a consequent quantum mechanical description of multiboson systems is presented, using properly normalized projector operators for overlapping multiparticle wave-packet states describing stimulated emission of bosons. Our new analytic result is that multiboson correlations generate *momentum-dependent* radius and intercept parameters even for *static* sources, as well as induce a special *directional dependence* of the correlation function. The effective *radius parameter* of the two-particle correlation function *is reduced for low values* and *enlarged for large values of the mean momentum* in the rare gas limiting case, as compared to the case when multiparticle symmetrization effects are neglected. For extended, hot, and rare gas of a few hundred pions, the reduction of the radius parameters at low momentum

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