

Quantum Geometry and Black Hole Entropy

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A “black hole sector” of nonperturbative canonical quantum gravity is introduced. The quantum black hole degrees of freedom are shown to be described by a Chern-Simons field theory on the horizon. It is shown that the entropy of a large nonrotating black hole is proportional to its horizon area. The constant of proportionality depends upon the Immirzi parameter, which fixes the spectrum of the area operator in loop quantum gravity; an appropriate choice of this parameter gives the Bekenstein-Hawking formula $S = A/4\ell_p^2$. With the *same choice* of the Immirzi parameter, this result also holds for black holes carrying electric or dilatonic charge, which are not necessarily near extremal. [S0031-9007(97)05183-1]

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The statistical mechanical origin of black hole entropy has drawn a great deal of attention recently (for reviews, see, for example, [1–3]). Most of the work based on string theory has focused on the extremal and near extremal cases. The purpose of this Letter is to introduce a new framework, based on nonperturbative quantum gravity [4,5], that enables one to treat general black holes in four dimensions.

The basic ideas can be summarized as follows. We first introduce a sector of the classical theory that corresponds to isolated, nonrotating black holes and find the associated phase space description. Then we quantize the resulting phase space. Finally, we isolate the quantum states that describe the geometry of the horizon. It is these degrees of freedom that account for the black hole entropy in our approach. We find that the statistical mechanical entropy of the black hole is proportional to its horizon area.

Recently, nonperturbative techniques have led to a quantum theory of geometry in which operators corresponding to lengths, areas, and volumes have discrete spectra. Of particular interest are the spin-network states associated with graphs in 3-space with edges labeled by spins $j = \frac{1}{2}, 1, \dots$, and vertices labeled by intertwining operators [6,7]. If a single edge punctures a 2-surface transversely, it contributes an area proportional to $\sqrt{j(j+1)}$ [8,9]. Over the last two years, this picture led to certain constructions which in turn inspired the present work. First, while working with a space-time region with boundary in Euclidean general relativity with nonzero cosmological constant, Smolin [10] was led to introduce gravitational surface states which could be identified with the states of the SU(2) Chern-Simons theory on a surface with punctures. Second, Rovelli [11], motivated by the work of Krasnov [12], estimated the number of spin-network states which endow a 2-sphere with a given, large area and applied this estimate to black hole horizons. Third, Krasnov [13] proposed to combine the

two sets of ideas by introducing certain boundary conditions on regions bounded by 2-spheres. Finally, Carlip’s [1] considerations of surface states in the context of three-dimensional black holes also played a suggestive role in the classical part of our treatment.

Let us begin with uncharged, nonrotating black holes. A sector of the classical phase space corresponding to an isolated, nonrotating black hole can be constructed as follows. Consider the manifold (with boundary) representing the asymptotic region of Fig. 1. We refer to the outer boundary as I and the inner boundary as \mathcal{H} . Our dynamical fields are a soldering form $\sigma_a^{AA'}$ for SL(2,C) spinors and an SL(2,C) connection A_{aA}^B [4,14]. (In a classical solution, $g_{ab} = \sigma_a^{AA'}\sigma_{bAA'}$ is the Lorentzian space-time metric and A_{aA}^B is the self-dual connection that operates only on unprimed spinors.) On I , fields are required to satisfy the standard asymptotically flat boundary conditions. The conditions on \mathcal{H} , on the other hand, are more subtle and will be discussed in detail elsewhere. The

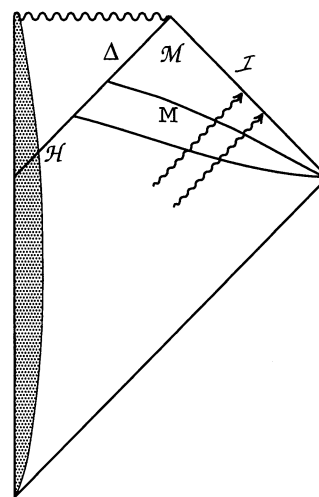


FIG. 1. Example of a spacetime of interest.

key requirements are as follows: (i) \mathcal{H} is a null surface with respect to the metric g_{ab} ; (ii) on a “finite patch” Δ of \mathcal{H} , the area of any cross section is a constant, A_S , the Weyl spinor is of Petrov types 2-2, and its only nonzero component, Ψ_2 , is given by $\Psi_2 = 2\pi/A_S$; and (iii) the 2-flats on Δ , orthogonal to the two principal null directions of the Weyl tensor, span 2-spheres, and the pullback of the connection A_a to these 2-spheres is real.

In what follows, we focus only on the part Δ of \mathcal{H} and the corresponding region \mathcal{M} of the spacetime (see Fig. 1). Roughly, condition (ii) implies that there is no gravitational radiation falling into Δ (i.e., the black hole is “isolated” there), while the first part of (iii) implies that it is “nonrotating.” The three conditions together imply that, on partial Cauchy surfaces (such as M) that intersect Δ in the preferred 2-spheres, the 2-spheres are marginally outer trapped surfaces. These boundary conditions have been extracted from the geometrical structure available at the Schwarzschild horizon. However, we do *not* require staticity and allow gravitational waves in the exterior region. Therefore, our phase space will be infinite dimensional; the boundary conditions are quite weak.

However, these boundary conditions are strong enough to imply that the variational principle is well defined. More precisely, we can add to the standard self-dual action [14] a surface term so that the total action is functionally differentiable and yields precisely the Einstein equations,

$$S(\sigma, A) = -\frac{i}{8\pi G} \int_{\mathcal{M}} \text{Tr}(\Sigma \wedge F) - \frac{i}{8\pi G} \frac{A_S}{4\pi} \times \int_{\Delta} \text{Tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right). \quad (1)$$

Here $\Sigma_{ab}^{AB} = 2\sigma_{[a}^{AA'} \sigma_{b]A'}^B$ while F_{abA}^B is the curvature of the connection A , and G is Newton’s constant. (Throughout we have set $c = \hbar = 1$.) Note that the required surface term is precisely the action of Chern-Simons theory. It is straightforward to cast the theory into Hamiltonian form. The basic phase space variables are the restrictions of Σ and A to the spatial hypersurface M with a boundary S . (Vector densities, dual to the pullback of Σ , are the familiar density weighted triads.) Unfortunately, the restriction to M of the self-dual connection A is a complex valued SU(2) connection and the functional analysis required to handle complex connections in quantum theory is not yet fully developed.

Therefore, at this stage, it is easier to make a transformation to real variables [15]. On M , A can be expressed in terms of real fields as $A_a = \Gamma_a - iK_a$, where Γ is the three-dimensional spin connection compatible with the triad field and K is the extrinsic curvature of M . This suggests [16] that we introduce real phase space variables ${}^\gamma A_a := \Gamma_a - \gamma K_a$ and ${}^\gamma \Sigma_{ab} := (1/\gamma)\Sigma_{ab}$, where γ is a positive real number known as Immirzi parameter. Then, our boundary conditions imply that the phase space con-

sists of real fields (${}^\gamma A_a, {}^\gamma \Sigma_{ab}$) that are asymptotically flat at infinity and satisfy the following condition at S :

$${}^\gamma \underline{F}_{ab}^{AB} = -\frac{2\pi\gamma}{A_S} {}^\gamma \underline{\Sigma}_{ab}^{AB}, \quad (2)$$

where underbars denote pullbacks to S . This in turn implies that the restriction of ${}^\gamma A$ to S yields a reducible connection, i.e., one satisfying $D_a r = 0$ for some “radial” internal vector r . Although it is not necessary to do so, for technical simplicity, we fix r on S using the SU(2) gauge freedom. Then the gauge group on the boundary is reduced to U(1) and only the r component of (2) is nontrivial.

On this manifestly real phase space, the symplectic structure derived from (1) is given by

$$\begin{aligned} \Omega|_{({}^\gamma A, {}^\gamma \Sigma)} &= ((\delta{}^\gamma A, \delta{}^\gamma \Sigma), (\delta{}^\gamma A', \delta{}^\gamma \Sigma')) \\ &= \frac{1}{8\pi G} \int_M \text{Tr}[\delta{}^\gamma \Sigma \wedge \delta{}^\gamma A' - \delta{}^\gamma \Sigma' \wedge \delta{}^\gamma A] \\ &\quad - \frac{k}{2\pi} \oint_S \text{Tr}[\delta{}^\gamma A \wedge \delta{}^\gamma A'], \end{aligned} \quad (3)$$

$$\text{where } k = \frac{A_S}{8\pi\gamma G} \quad (4)$$

is later identified with the level of the Chern-Simons theory. Up to a numerical coefficient, k is simply the area of the horizon of black hole measured in the units of Planck area $\ell_P^2 = G$ [13]. Note that, in addition to the familiar volume term, the symplectic structure has a surface term which coincides with the symplectic structure of the Chern-Simons theory.

The theory has three sets of first class constraints. A careful analysis shows that they generate the following gauge transformations: (i) SU(2) internal rotations that reduce to U(1) rotations preserving a fixed vector r on the boundary S ; (ii) spatial diffeomorphisms that leave S invariant; and (iii) canonical transformations generated by the scalar constraint with lapse fields approaching zero at spatial infinity and on S . Somewhat surprisingly, it turns out that condition (2), the pullback to S of the type 2-2 requirement, ensures full gauge invariance on the boundary. Without it, as in the case of the scalar constraint, only the internal rotations whose generators vanish on S could be regarded as gauge.

It is intuitively clear that not all the degrees of freedom described by fields ${}^\gamma A, {}^\gamma \Sigma$ are relevant to the problem of black hole entropy. In particular, there are “volume” degrees of freedom in the theory corresponding to gravitational waves far away from Δ which should not contribute. It has often been argued (see, e.g., [3] and references therein) that it is the degrees of freedom “living on the horizon” that should account for the entropy. We adopt this viewpoint in our approach.

In the classical theory that we have described, the volume and surface degrees of freedom cannot be separated: all fields on S are determined by fields in the interior of

M by continuity. However, in the quantum theory, the fields describing geometry become discontinuous in certain precise sense [6], and the fields on S are no longer determined by fields in M ; in this case there are independent degrees of freedom “living” on the boundary. These surface degrees of freedom are the ones that account for black hole entropy in our approach.

To quantize the theory, we first construct a Hilbert space \mathcal{H}^V of volume states and a Hilbert space \mathcal{H}^S of “surface” states, and then impose constraints on $\mathcal{H}^V \otimes \mathcal{H}^S$ to obtain the space of physical states. We take \mathcal{H}^V to consist of certain square-integrable functions on the space of generalized SU(2) connections [6] on M modulo gauge transformations that are the identity on S . The form of the Hilbert space \mathcal{H}^S of surface states is motivated by the fact that in the quantum theory we wish to impose the boundary condition (2) as an operator equation. That is, given a spin network state Ψ_V in \mathcal{H}^V and a state Ψ_S in \mathcal{H}^S , the quantum version of the r component of Eq. (2) should read

$$\left(1 \otimes \frac{A_S}{2\pi\gamma} \hat{F}_{ab} \cdot r + \hat{\Sigma}_{ab} \cdot r \otimes 1\right) \Psi_V \otimes \Psi_S = 0. \quad (5)$$

The structure of this equation implies that Ψ_V and Ψ_S should be eigenstates of $\hat{\Sigma}_{ab} \cdot r$ and $\hat{F}_{ab} \cdot r$, respectively. Now, the “polymer nature” of quantum geometry in M implies that eigenvalues of $\hat{\Sigma}_{ab} \cdot r$ are distributional, given by [9]

$$8\pi\ell_p^2 \sum_i j_i \delta^2(x, p_i) \eta_{ab} \quad (6)$$

for some points p_i on S , where j_i are half-integers, δ^2 is the delta distribution on S , and η_{ab} the Levi-Civita density on S . Therefore, (5) implies that the surface states Ψ_S have support only on generalized connections that are everywhere flat except at a finite number of points p_i . It turns out that such generalized connections can be identified with ordinary connections with distributional curvature. Since the surface symplectic structure is that of Chern-Simons theory, for any fixed choice

$$\mathcal{P} = \{(p_1, j_{p_1}), \dots, (p_n, j_{p_n})\} \quad (7)$$

of points in S labeled by spins, we wish \mathcal{H}^S to have a subspace $H_{\mathcal{P}}^S$ given by the space of states of U(1) Chern-Simons theory on a sphere with punctures p labeled by spins j_p . The total space \mathcal{H}^S is the direct sum of these subspaces. The prequantization condition of the Chern-Simons theory requires that the “level” k defined in (4) be an integer.

Note now that $(k/2\pi)\hat{F}$ is the generator of internal rotations in Chern-Simons theory. Thus, the meaning of (5) turns out to be rather simple: it ensures that the volume and surface states are “coupled” in precisely the correct way so that the total state is invariant under U(1) internal rotations at S . The remaining constraints require

that the states be invariant under diffeomorphisms of M that leave S invariant and under motions generated by the Hamiltonian constraint smeared with any lapse field that vanishes at S and at infinity. Thus, the following physical picture emerges. For each set \mathcal{P} of finitely many punctures p labeled by spins j_p , there is a subspace $H_{\mathcal{P}}^V$ of volume states having a basis given by open spin networks whose edges intersect S only at these punctures, where they are labeled by the spins j_p . Similarly there is a subspace $H_{\mathcal{P}}^S$ consisting of quantum states of U(1) Chern-Simons theory on the punctured surface S . The total physical Hilbert space is given by

$$\mathcal{H}_{\text{phy}} = \frac{\bigoplus_{\mathcal{P}} [\mathcal{H}_{\mathcal{P}}^V \otimes \mathcal{H}_{\mathcal{P}}^S]}{\text{gauge}},$$

where “gauge” means internal SU(2) rotations that reduce to U(1) on S , diffeomorphisms preserving S , and the motions generated by the Hamiltonian constraint. The quotient by diffeomorphisms identifies any two Hilbert spaces associated with sets \mathcal{P} that can be mapped into each other by a diffeomorphism on S . Thus, what matters is only the spins labeling punctures, not the locations of individual punctures. Unfortunately, we do not have yet a complete control over the quantum Hamiltonian constraint, despite the recent progress on this front [17]. To proceed, we make a rather weak assumption about the quantum dynamics: namely, generically there is at least one solution of this constraint in $\mathcal{H}_{\mathcal{P}}^V \otimes \mathcal{H}_{\mathcal{P}}^S$ for any set \mathcal{P} of punctures labeled by spins.

We are not interested in this full Hilbert space since it includes, e.g., states of gravitational waves far away from Δ . Rather, we wish to consider only states of the horizon of a black hole with large area A_S . Thus we trace over the volume states to construct a density matrix ρ_{bh} describing a maximal-entropy mixture of surface states for which the area of the horizon lies in the range $A_S \pm \ell_p^2$. The statistical mechanical black hole entropy is then given by $S_{\text{bh}} = -\text{Tr} \rho_{\text{bh}} \ln \rho_{\text{bh}}$. As usual, this can be computed simply by counting states: $S_{\text{bh}} = \ln N_{\text{bh}}$ where N_{bh} is the number of Chern-Simons surface states satisfying the area constraint.

Fortunately, the eigenvalues of the area operator are explicitly known. For the case now under consideration, they are given by [8,9]

$$a_{\mathcal{P}} = 8\pi\gamma\ell_p^2 \sum_p \sqrt{j_p(j_p + 1)}, \quad (8)$$

where j_p are the spins labeling the punctures as in (7). Thus, $N_{\text{bh}} = \sum_{\mathcal{P}} N_{\mathcal{P}}$, where the summation extends over those sets of labeled punctures \mathcal{P} for which $A_S - \ell_p^2 \leq a_{\mathcal{P}} \leq A_S + \ell_p^2$ and $N_{\mathcal{P}}$ is the dimension of the Hilbert space of the Chern-Simons theory associated with \mathcal{P} . Note that even though $A_S = 8\pi k \ell_p^2$ where k is an integer, for large k there exist many $a_{\mathcal{P}}$ satisfying the area constraint. Using the fact (from Chern-Simons theory)

that for a large number of punctures the dimension $N_{\mathcal{P}}$ of $\mathcal{H}_{\mathcal{P}}^S$ grows as

$$N_{\mathcal{P}} \sim \prod_{j_p \in \mathcal{P}} (2j_p + 1), \quad (9)$$

it is straightforward to calculate the entropy. For large A_S it is given by

$$S_{\text{bh}} = \frac{\gamma_0}{4\ell_P^2 \gamma} A_S, \quad \gamma_0 = \frac{\ln 2}{\pi \sqrt{3}}, \quad (10)$$

where the appearance of γ can be traced back directly to the formula for the eigenvalues of the area operator, (8). Thus, in the limit of large area, the entropy is proportional to the area of the horizon. If we set $\gamma = \gamma_0$, the statistical mechanical entropy is given precisely by the Bekenstein-Hawking formula.

Are there independent checks on this preferred value? The answer is in the affirmative. One can carry out this calculation for Reissner-Nordstrom as well as dilatonic black holes. *A priori* it could have happened that, to obtain the Bekenstein-Hawking value, one would have to readjust the Immirzi parameter for each value of the electric or dilatonic charge. This does *not* happen. The entropy is still given by (10) and hence by the Bekenstein-Hawking value when $\gamma = \gamma_0$.

To summarize, we first introduced a black hole sector of the gravitational phase space and then quantized it using the by now well-developed framework of nonperturbative quantum gravity together with results from Chern-Simons theory. We found that the entropy is proportional to the area irrespective of the value of the Immirzi parameter γ and that a single choice of γ yields the Bekenstein-Hawking coefficient irrespective of the parameters labeling the nonrotating black hole.

We conclude with a few remarks:

(i) One can show [16] that different values of γ correspond to different “sectors,” that is, unitarily inequivalent representations of the canonical commutation relations. The spectrum of the area operator is different in each representation. As usual in such situations, the “correct” sector can only be singled out by additional input. The Bekenstein-Hawking calculation can be regarded as serving this purpose. However, the full significance of γ is yet to be understood.

(ii) A detailed calculation shows that the states which dominate the counting correspond to punctures all of which have labels $j = 1/2$. Thus, there is a curious similarity between our detailed results and John Wheeler’s “it from bit” picture [18] of the origin of black hole entropy.

(iii) So far, we have only considered nonrotating black holes. However, the basic ideas underlying this framework apply also to the rotating case.

(iv) Our approach provides only an “effective” description of a quantum black hole, for we first isolated a black hole sector classically and then quantized that sector. The issue of extracting this sector from a complete theory of quantum gravity is yet to be explored. Nonetheless, it is rather striking that subtle results from quite different areas—classical general relativity, quantum geometry, and Chern-Simons theory—fit tightly without a mismatch to provide a coherent picture of the microstates of a black hole. The detailed implications of this picture for the black hole evaporation process are now being explored [19].

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