Statistical Evolution of Chaotic Fluid Mixing

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We describe a new constitutive theory for two-phase flow models of chaotic mixing layers, which form as two incompressible fluids interpenetrate. This theory is compatible with arbitrary velocities of the edges of a mixing layer, and it gives analytic solutions for the distribution of fluid variables across the layer in terms of these velocities. Our results are in agreement with all available data from planar Rayleigh-Taylor instability experiments. The model that we discuss can be embedded in a larger system of two-phase flow equations in order to predict other important physical quantities, such as the fluid pressures and internal energies in compressible mixing. [S0031-9007(97)04668-1]

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Mixing layers form in the late evolution stage of unstable fluid interfaces [1], for example, in the acceleration-driven Rayleigh-Taylor (RT) and Richtmeyer-Meshkov (RM) instabilities and in the shear-driven Kelvin-Helmholtz (KH) instability. They are of fundamental importance in natural phenomena such as supernova explosions [2], and in technological applications such as inertial confinement fusion [3].

In this Letter we describe a new constitutive theory for the average interface velocity in a chaotic mixing layer formed from interpenetrating incompressible fluids. The fundamental quantity is the propagation speed v^* which appears in a hyperbolic partial differential equation (PDE) for the fluid volume fraction. Our main results are: (a) A theory relating v^* to a convex linear combination of the individual mean fluid velocities v_1 and v_2. (b) A procedure for inferring the distributions of v_1, v_2, and v^* from measured volume fraction profiles, which thus provides a means to directly measure v^* and determine its dependence on v_1 and v_2 using currently available experimental techniques. (c) A fractional linear model for the coefficients in the linear combination. For an expanding mixing layer in the self-similar regime of RT instability, this model predicts a linear volume fraction profile at small to moderate Atwood number, in agreement with all available experimental data. (d) A prediction for the expansion ratio α_2/α_1 of RT mixing layers at all Atwood numbers, which provides the closest agreement with experimental data to date.

Consider an infinite ensemble of chaotic mixing layers that form from the unstable growth of initially small perturbations at an interface between two fluids. We assume that the statistics of the ensemble are translationally invariant in the x and y directions, and that any external acceleration or impulse, as occurs in the RT and RM problems, is directed along the z axis. Therefore all ensemble-averaged quantities depend only on z and t. This assumption allows fluid motion in a direction tangential to the mixing layer, as occurs in the KH problem, but note that the shearing motion must be translationally invariant in the x and y directions. Instabilities driven by forces oblique to the fluid interface are not included in this framework, except perhaps in a local approximation.

A hyperbolic PDE for the volume fraction follows from the steps of ensemble averaging the kinematic constraint at material interfaces in the preaveraged flow (as described by Drew [4]) and imposing translational symmetry [5],

$$\frac{\partial \beta_k}{\partial t} + v^* \frac{\partial \beta_k}{\partial z} = 0\,, \tag{1}$$

where $\beta_k(z,t)$ is the volume fraction of fluid k. We now consider incompressible mixing, for which the setup and notation are shown in Fig. 1. The ensemble average of the continuity condition $\nabla \cdot \mathbf{v} = 0$ within each fluid, with

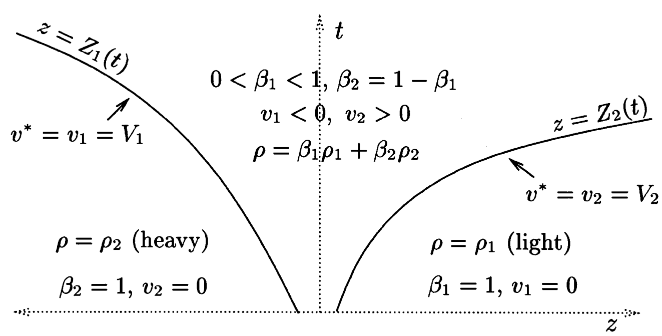

FIG. 1. Incompressible two-phase mixing in the (z,t) plane. The two curves are the trajectories of the mixing zone edges. The lower (upper) edge is the limit of vanishing β_1 (β_2), and it corresponds to the tip of the frontier portion of light (heavy) fluid in the *preaveraged* flow.

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translational symmetry imposed, is [6]

$$\frac{\partial \beta_k v_k}{\partial z} = v^* \frac{\partial \beta_k}{\partial z} . \tag{2}$$

There are now three independent equations, namely, Eq. (1) for either k and (2) for $k = 1, 2$, for the four unknowns β_1, v_1, v_2, and v^*; β_2 is trivially eliminated using the identity $\beta_1 + \beta_2 = 1$. As a closure hypothesis, we propose to replace v^* by a function of v_1 and v_2 and additional variables of the problem that are spatially dimensionless, e.g., t and β_k. This idea is the unique aspect of our approach. Below, we show that this general assumption constrains v^* to be a convex linear combination of v_1 and v_2.

Mathematically, the requirement on closure is that the number of independent equations equal the number of unknowns. The proof of independence is the unique solvability of the resulting equations. All solution steps used here are unique, and hence any $v^* = v^*(t, \beta_1, v_1, v_2)$ closure is correct on a mathematical basis. Rewriting Eq. (2) to yield $v^* = d\beta_k v_k / d\beta_k$, it is clear that the closure model for v^* imposes a single functional relation between the otherwise unconstrained variables β_k and v_k.

The absence of the single-phase pressures p_k in the v^* closure and in the boundary conditions for the mixing zone edges decouples the β_k and v_k equations from the momentum equations and renders them soluble in our model. Because equilibrated pressure closures are often used in multiphase flow analyses [7–11], we emphasize that any closure [$p_2 = p_1$ vs $v^* = v^*(t, \beta_1, v_1, v_2)$] represents a restriction of generality, and as in all thermodynamic modeling, its validity can only be assessed through comparison to specific flow regimes. See [5] for a quantitative validation of a particular v^* closure for two-dimensional compressible RT mixing. In RT mixing under a constant acceleration, the pressures do not equilibrate ($p_2 \neq p_1$), according to simple physical arguments given in [6,12]. Nonequilibration of pressure in RT instability is especially easy to understand in the case of an infinite density ratio, for then p_1 is identically zero everywhere (since phase 1 is a vacuum) while p_2 cannot possibly satisfy this constraint throughout the mixing region. Our purpose here is not to emphasize the deficiencies of single-pressure two-phase flow models (which have been discussed previously [5,6,13–15]), but rather to provide an alternative approach which is compatible with a two-pressure formulation [5,12].

Consider the problem of inferring v^* from experimental data for incompressible two-fluid mixing. Equation (1) describes the propagation of surfaces of constant volume fraction. Current experimental techniques are adequate for determining the distribution of volume fraction across the mixing layer at various times. It is clear how to determine v^* as a function of volume fraction in this context: measure two β_1 (or β_2) profiles separated by a short time interval; v^* at any β_1 is then the change of the height where β_1 occurs divided by the time interval.

The data for v^* as a function of β_1 directly determines the velocity profiles. Integrating Eq. (2) for $k = 1$ over β_1, we have

$$\beta_1 v_1 = \int_0^{\beta_1} v^* \, d\beta_1' . \tag{3}$$

Summing Eq. (2) over k and using $\beta_1 + \beta_2 = 1$, we get $\partial(\beta_1 v_1 + \beta_2 v_2)/\partial z = 0$. The solution to this ordinary differential equation (ODE) which satisfies the boundary condition that $v_1 = 0$ ($v_2 = 0$) at the upper (lower) wall of a finite but large domain is

$$\beta_1 v_1 + \beta_2 v_2 = 0 . \tag{4}$$

To summarize, volume fraction profiles measured over short time intervals determine the convective speed v^* of the volume fraction mode. The solutions for v_1 and v_2 follow from Eqs. (3) and (4). By this procedure, one has the means to test any constitutive law relating v^* to t, β_1, v_1, and v_2.

We now show that our general assumption regarding the dependence of v^* constrains this quantity to be a convex linear combination of v_1 and v_2, i.e.,

$$v^* = \mu_2^v v_1 + \mu_1^v v_2 , \tag{5}$$

with $\mu_k^v \geq 0$ and $\mu_1^v + \mu_2^v = 1$. This fact is a consequence of the following proposition.

Proposition.—Let $U(v_1, v_2)$ be a smooth real-valued function which is both scale and translation invariant, by which we mean that $U(av_1, av_2) = aU(v_1, v_2)$ for $a \geq 0$ and $U(v_1 + b, v_2 + b) = U(v_1, v_2) + b$ for all real b. Assume also that U is non-negative if both v_1 and v_2 are. Then U is a convex linear combination of v_1 and v_2.

Proof.—Applying first translation invariance and then scale invariance, we obtain

$$\begin{aligned} U(v_1, v_2) &= v_2 + U(v_1 - v_2, 0) \\ &= v_2 + |v_1 - v_2| U(\mathrm{sgn}(v_1 - v_2), 0) . \end{aligned}$$

Thus U is uniquely determined by the two numbers $U(\pm 1, 0)$. Differentiating this formula in the two regions $v_1 > v_2$ and $v_1 < v_2$ we obtain $\partial U / \partial v_1 = U(1, 0)$ and $\partial U / \partial v_1 = -U(-1, 0)$. By smoothness of U, both identities must hold on the common boundary of these two regions, where $v_1 = v_2$. From this fact, we conclude that $-U(-1, 0) = U(1, 0)$ and that

$$U(v_1, v_2) = v_2 + (v_1 - v_2) U(1, 0) .$$

It follows that U is a linear combination of v_1 and v_2, and the respective coefficients $U(1, 0)$ and $1 - U(1, 0)$ sum to unity. From the non-negativity assumption, we see that $U(1, 0)$ and $1 - U(1, 0)$ are each non-negative, so that the linear combination is convex. $\square$

In the application of this proposition to the closure for v^*, we obtain scale invariance from dimensional reasoning and translation invariance from Galilean frame invariance of the original equations. The positivity and smoothness assumptions are additional requirements of a very reasonable nature. The additional spatially dimensionless arguments we consider for μ_k^v are β_k and t. Consistency of v^* with the picture in Fig. 1 leads to the boundary conditions

$$\mu_k^v(t, \beta_k = 0) = 0, \qquad \mu_k^v(t, \beta_k = 1) = 1\,. \tag{6}$$

We now use the constitutive law (5) to solve for the fluid velocities and volume fractions. Using Eqs. (4) and (5) to eliminate one of the velocities from Eq. (2), we obtain

$$-\frac{1}{v_k}\frac{dv_k}{d\beta_k} = \left[\frac{\mu_k^v(t,\beta_k)}{\beta_k} - \frac{\mu_{k'}^v(t,\beta_{k'})}{\beta_{k'}} + \frac{1}{\beta_{k'}}\right], \tag{7}$$

where $k' = 3 - k$. The solution to this ODE for the boundary conditions shown in Fig. 1 is

$$v_k = V_k \beta_{k'} e^{-F_k(t,\beta_k)}, \tag{8}$$

where

$$F_k(t,\beta_k) = \int_0^{\beta_k}\left[\frac{\mu_k^v(t,\phi_k)}{\phi_k} - \frac{\mu_{k'}^v(t,\phi_{k'})}{\phi_{k'}}\right] d\phi_k\,. \tag{9}$$

In this integration the relation $\phi_k + \phi_{k'} = 1$ holds. From Eqs. (2) and (7), we have

$$v^* = [\beta_{k'}\mu_{k'}^v(t,\beta_{k'}) - \beta_k \mu_k^v(t,\beta_k)]\frac{v_k}{\beta_{k'}}\,. \tag{10}$$

The RHS of this expression must give the same v^* for both $k = 1$ and $k = 2$. It is easy to show [12] that this condition is satisfied if and only if

$$-\frac{V_2(t)}{V_1(t)} = e^{-F_1(t,1)}. \tag{11}$$

As a specific choice of constitutive law, we propose the linear fractional form

$$\mu_k(t,\beta_k) = \frac{a_k(t)\beta_k + d_k(t)\beta_{k'}}{c_k(t)\beta_k + b_k(t)\beta_{k'}}\,, \tag{12}$$

for $k = 1, 2$, where the a_k, b_k, c_k, and d_k are time-dependent functions to be determined. The relations $d_k = 0$ and $c_k = a_k$ follow from Eqs. (6). Also, μ_k is invariant under an arbitrary scaling of both numerator and denominator in Eq. (12), so we can set either a_k or b_k arbitrarily. We choose $a_k(t) = |V_k(t)|$. The remaining unknowns b_1 and b_2 are determined from frame invariance ($\mu_1^v + \mu_2^v = 1$) and Eq. (11). The unique solution is $b_k(t) = |V_{k'}(t)|$; hence

$$\mu_k(t,\beta_k) = \frac{|V_k|\beta_k}{|V_1|\beta_1 + |V_2|\beta_2}\,. \tag{13}$$

Equations (8), (9), (10), and (13) give v^* as a function of t and β_k,

$$v^*(t,\beta_k) = \frac{|V_1V_2|\,(|V_1|\beta_1^2 - |V_2|\beta_2^2)}{(|V_1|\beta_1 + |V_2|\beta_2)^2}\,. \tag{14}$$

Solving the interface equation (1) by the method of characteristics, we get an implicit equation for the volume fraction profile,

$$z(\beta_k, t) = z(\beta_k, 0) + \int_0^t v^*(s,\beta_k)\, ds\,, \tag{15}$$

where the integrand is provided by Eq. (14). To summarize, Eqs. (8), (9), (13), (14), and (15) give the distributions of volume fractions and velocities across the mixing layer in terms of the trajectories of the edges.

In the special case that V_2/V_1 is independent of t, then so is μ_k. If, in addition, the mixing zone expands outward [i.e., $(-1)^k V_k > 0$ for all t], then there is a scale-invariant solution, where all lengths in the problem scale with the given time dependence of the edge displacements. One example is RT mixing under a constant acceleration $g > 0$, for which $Z_k(t) = (-1)^k \alpha_k A g t^2$, where α_1 and α_2 are positive constants which depend on the Atwood ratio $A = (\rho_2 - \rho_1)/(\rho_2 + \rho_1)$. The ratio $|V_2/V_1| = \alpha_2/\alpha_1$ is constant in this problem, and Eqs. (14) and (15) give the scale-invariant solution for the volume fraction profile,

$$\frac{z}{Agt^2} = \frac{\alpha_1\alpha_2(\beta_1^2\alpha_1 - \beta_2^2\alpha_2)}{(\alpha_1\beta_1 + \alpha_2\beta_2)^2}\,. \tag{16}$$

The scale-invariant solution for the velocities v_k follows by evaluation of Eq. (8),

$$\frac{v_k}{2Agt} = (-1)^k\frac{\alpha_1\alpha_2\beta_{k'}}{\alpha_1\beta_1 + \alpha_2\beta_2}\,. \tag{17}$$

When the expansion ratio $\alpha_2/\alpha_1 = 1$, Eq. (16) implies that the volume fraction varies linearly across the mixing zone. As seen in Fig. 2, α_2/α_1 increases very slowly with increasing A, i.e., $\alpha_2/\alpha_1 \approx 1$ up to moderate A. Thus Eq. (16) predicts nearly linear profiles for small to moderate A, in agreement with all currently available experimental data for planar RT instability [7,16,17]. The correct shape of the volume fraction profiles at large A has not been adequately established.

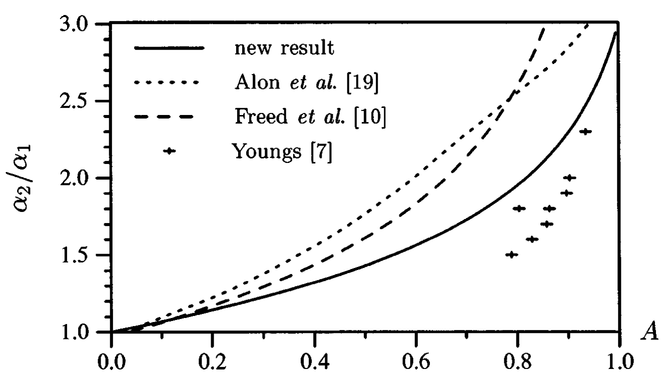

FIG. 2. The expansion ratio α_2/α_1 of the mixing zone as a function of the Atwood ratio $A = (\rho_2 - \rho_1)/(\rho_2 + \rho_1)$.

Equations (16) and (17) give a two-parameter family of self-similar solutions for RT mixing. To obtain a prediction of α_2/α_1 from this model therefore requires imposing an additional physical constraint on the problem. Here we discuss the possibility that the mixing layer center of mass (c.m.) position is universally stationary in RT mixing.

The c.m. position $Z(t)$ is given by

$$Z(t) = \frac{1}{M(t)} \int_{Z_1(t)}^{Z_2(t)} (\beta_1 \rho_1 + \beta_2 \rho_2) z \, dz ,$$

where the mixing layer mass $M(t) = M(0) + \rho_1[Z_2(t) - Z_2(0)] + \rho_2[Z_1(0) - Z_1(t)]$. In the self-similar regime of RT mixing, one can show after some algebra that

$$\hat{M}\hat{Z} = \frac{1}{4}(1 + A)(\alpha_2^2 - \alpha_1^2) - A \int_{-\alpha_1}^{\alpha_2} \beta_1 \hat{z} \, d\hat{z} , \quad (18)$$

where $\hat{Z} = Z/Agt^2$, $\hat{z} = z/Agt^2$, and $\hat{M} = M/(\rho_2 - \rho_1)gt^2$.

It is clear that $\hat{Z}$ should vanish in the symmetric limit ($A \to 0$). For $\hat{Z}$ to remain zero as A increases, the spikes (the penetrating portions of heavy fluid) must become increasingly thin to cancel their favored mass, but they need not become infinitesimally thin as $A \to 1$, as there is always heavy fluid between the bubbles (the penetrating portions of light fluid). This qualitative behavior has been confirmed in numerical simulations (see, for example, Ref. [18]). Thus $\hat{Z} = 0$ has a clear physical interpretation and appears to be plausible in an approximate sense. Note that we are considering the center of mass of the mixing layer and not the center of mass of the entire fluid system.

Imposing $\hat{Z} = 0$ on the self-similar β_k profile given implicitly by Eq. (16) constrains α_2/α_1 to be a function of A. This relation is displayed as the solid curve in Fig. 2. For comparison, the figure includes the measurements of Youngs [7] and theoretical predictions based on the two-phase model of Freed *et al.* [10] and the statistical merger model of Alon *et al.* [19]. Freed *et al.* actually provide two different analytical approximations for α_2/α_1, one for small A and one for large A. Because these approximations do not match at any A, we avoid the problem of blending them and show only the large A approximation, which pertains to the more interesting region of Fig. 2 while still being reasonably accurate at small A.

The expansion ratio is sensitive to where one terminates the mixing zone. All of the curves shown in Fig. 2 are for a precise (0%) cutoff, consistent with the method used by Read and Youngs [7,20]. Freed *et al.* assumed a 5% cutoff (i.e., the mixing zone edges were considered to be where $\beta_1 = 0.05$ and 0.95) when comparing their theoretical expansion ratios to the same experimental data [10]. Increasing the cutoff criterion reduces the expansion ratio; a 5% cutoff would lower the solid curve in Fig. 2 enough to make it agree with the experimental data to within the probable uncertainty in the measurements. For any choice of cutoff, the model prediction clearly gives the closest agreement with experimental data to date.

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[1] J. Glimm and D. H. Sharp, in *Stochastic Partial Differential Equations: Six Perspectives, Mathematical Surveys and Monographs,* edited by R. A. Carmona and B. L. Rozovskii (American Mathematical Society, Providence, 1997).
[2] D. Arnett, B. Fryxell, and E. Muller, Astrophys. J. **341**, L63 (1989).
[3] J. D. Kilkenny *et al.*, Phys. Plasmas **1**, 1379 (1994).
[4] D. A. Drew, Annu. Rev. Fluid Mech. **15**, 261 (1983).
[5] Y. Chen, J. Glimm, D. H. Sharp, and Q. Zhang, Phys. Fluids **8**, 816 (1996).
[6] J. Glimm, D. Saltz, and D. H. Sharp, Phys. Lett. A **222**, 171 (1996).
[7] D. L. Youngs, Physica (Amsterdam) **37D**, 270 (1989).
[8] F. Harlow and A. Amsden, J. Comput. Phys. **18**, 440 (1975).
[9] D. L. Youngs, Physica (Amsterdam) **12D**, 32 (1984).
[10] N. Freed, D. Ofer, D. Shvarts, and S. Orszag, Phys. Fluids A **3**, 912 (1991).
[11] U. Alon and D. Shvarts, in *Proceedings of the Fifth International Workshop on Compressible Turbulent Mixing*, edited by R. Young, J. Glimm, and B. Boston (World Scientific, Singapore, 1996).
[12] J. Glimm, D. Saltz, and D. H. Sharp (to be published).
[13] V. H. Ransom and D. L. Hicks, J. Comput. Phys. **53**, 124 (1984).
[14] H. B. Stewart and B. Wendroff, J. Comput. Phys. **56**, 363 (1984).
[15] D. Holm and B. Kupershmidt, Phys. Lett. A **106**, 165 (1984).
[16] D. M. Snider and M. J. Andrews, Phys. Fluids **6**, 3324 (1994).
[17] M. B. Schneider, G. Dimonte, and B. Remington (to be published).
[18] C. L. Gardner *et al.*, Phys. Fluids **31**, 447 (1988).
[19] U. Alon, J. Hecht, D. Ofer, and D. Shvarts, Phys. Rev. Lett. **74**, 534 (1995).
[20] K. I. Read, Physica (Amsterdam) **12D**, 45 (1984).