Analysis of Random Cascades Using Space-Scale Correlation Functions

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(Received 26 June 1997)

We introduce a formalism that allows us to study space-scale correlations in multiscale processes. This method, based on the wavelet transform, is particularly well suited to study multiplicative random cascade processes for which the correlation functions take a very simple expression. This two-point space-scale statistical analysis is illustrated on some pedagogical examples and then applied to fully developed turbulence data. [S0031-9007(97)05105-3]

PACS numbers: 47.27.Eq, 02.50.–r, 47.27.Jv, 47.53.+n

Since the pioneering works of Mandelbrot [1], fractal concepts have become a standard tool to describe multiscale phenomena. The multifractal formalism introduced in the context of high Reynolds number turbulence [2] has been used extensively in many areas ranging from the study of strange attractors arising in chaotic situations [3,4] to nonequilibrium growth processes [5]. From a statistical point a view, this description amounts to characterizing an irregular field (e.g., a probability measure or a fractal signal) by the relative contribution of singularities of different strengths. More precisely, the so-called $f(\alpha)$ singularity spectrum [2,3] provides, at each scale, an estimate of the (log) number of points where the (log) amplitude of the field is α. The notion of *cascade* is widely used in the fractal literature and is often considered the paradigm of multifractal objects [3,4,6,7]. It refers to a self-similar process whose properties are defined multiplicatively from coarse to fine scales. Several solvable cascade models have been proposed to fit experimental observations. In the context of turbulence, the energy transfer from large eddies to smaller ones has been pictured by Richardson [8] and further developed by Kolmogorov and Obukhov [9], as a cascade process [4,6,10–12]. More recently, Castaing *et al.* [13] proposed to account for the probability distribution function (pdf) of the velocity increments at a given scale a, $\delta v_a = v(x+a) - v(x)$, as a weighted sum of dilated versions of the pdf at a coarser scale L:

$$P_a(\delta v) = \int_{-\infty}^{+\infty} G_{aL}(u) e^{-u} P_L(e^{-u}\delta v)\, du\,. \qquad (1)$$

This equation suggests that, for any decreasing sequence of scales $(a_1, \ldots, a_n)$, the kernel G satisfies the composition law $G_{a_n a_1} = G_{a_n a_{n-1}} \otimes \cdots \otimes G_{a_2 a_1}$ ($\otimes$ denotes the convolution product). It is then tempting to write an increment δv_a at a fine scale a as resulting from a random cascade initiated at the large scale L:

$$\delta v_a \equiv \prod_{i=1}^{n} W_{a_{i+1},a_i} \delta v_L \qquad (W_{a_{i+1},a_i} > 0)\,, \qquad (2)$$

where $\ln(W_{a_{i+1},a_i})$ are independent random variables of law G_{a_{i+1},a_i}. However, let us note that, in the same way that the $f(\alpha)$ spectrum provides rather poor information about the nature of the underlying process, Eq. (1) is a necessary but not sufficient condition for the existence of a cascade like Eq. (2). In this Letter, our goal is to show that one can go deeper in fractal analysis by studying correlation functions in both space and scales using the wavelet transform (WT). This "two-point" statistical analysis proves to be particularly well adapted for studying cascade processes. We illustrate our method on different well known stochastic models before applying it to turbulence data.

Correlations in multifractals have already been experienced in the literature [14]. However, all of these studies rely upon the computation of the scaling behavior of some partition functions involving different points; they thus mainly concentrate on spatial correlations of the local singularity exponents. Our approach is different since it does not focus on (nor suppose) any scaling property, but rather consists in studying the correlations of the *logarithms* of the amplitude of a space-scale decomposition of a signal. For that purpose we use a natural tool to perform space-scale analysis, namely the wavelet transform. The WT has already proven to be a powerful tool for multifractal analysis of functions [15]. Let us recall that the WT of a function f is defined as [16]:

$$T_\psi[f](x,a) = \frac{1}{a} \int_{-\infty}^{+\infty} f(y) \psi\left(\frac{x-y}{a}\right) dy\,, \qquad (3)$$

where x is the space position, a (>0) the scale, and ψ the analyzing wavelet. Note that for $\psi(x) = \delta(x-1) - \delta(x)$, $T_\psi[f](x,a)$ is nothing but $\delta f_a(x)$, the increment of f over a distance a. If ψ is oscillating and regular enough, one can show that the transformation (3) is invertible and conserves energy [16]. More specifically, if $\chi(x)$ is a bump function such that $\|\chi\|_1 = 1$, then by taking $\sigma^2(x,a) = a^{-2} \int \chi((x-y)/a) |T_\psi(y,a)|^2\, dy$, one has

$$\|f\|_2^2 = \int\int \sigma^2(x,a)\, dx\, da\,, \qquad (4)$$

and thus $\sigma^2(x,a)$ can be interpreted as the local space-scale energy density of f [17]. Since $\sigma^2(x,a)$ is a positive

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quantity, we can define the *magnitude* of the field f at point x and scale a as

$$\omega(x,a) = \frac{1}{2}\ln\sigma^2(x,a). \tag{5}$$

Our aim is to show that a cascade process can be studied through the correlations of its space-scale magnitudes:

$$C(x_1,x_2,a_1,a_2) = \overline{\tilde{\omega}(x_1,a_1)\tilde{\omega}(x_2,a_2)}, \tag{6}$$

where the overline stands for ensemble average, and $\tilde{\omega}$ stands for the centered process $\omega - \overline{\omega}$.

Cascade processes can be defined in various ways. Periodic wavelet orthogonal bases [16] provide a general framework in which they can be constructed easily [18,19]. Let us consider the following wavelet series:

$$f(x) = \sum_{j=0}^{+\infty}\sum_{k=0}^{2^j-1} c_{j,k}\psi_{j,k}(x), \tag{7}$$

where the set $\{\psi_{j,k}(x) = 2^{j/2}\psi(2^jx - k)\}$ is an orthonormal basis of $L^2([0,L])$ and the coefficients $c_{j,k}$ correspond to the WT of f at scale $a = L2^{-j}$ [L is the "integral" scale that corresponds to the size of the support of $\psi(x)$] and position $x = ka$. The above sampling of the space-scale plane defines a dyadic tree [16]. If one indexes by a dyadic sequence $\{\epsilon_1,\ldots,\epsilon_j\}$ ($\epsilon_k = 0$ or 1) each of the 2^j nodes at depth j of this tree, the cascade is defined by the multiplicative rule: $c_{j,k} = c_{\epsilon_1,\ldots,\epsilon_j} = c_0\prod_{i=1}^{j} W_{\epsilon_i}$. The law chosen for the weights W (accounting for their possible correlations) determines the nature of the cascade and the multifractal (regularity) properties of f [18]. From the above multiplicative structure, if one assumes that there is no correlation between the weights at a given cascade step [20], then it is easy to show that for $a_p = L2^{-j_p}$ and $x_p = k_p a_p$ ($p = 1$ or 2), the correlation coefficient is nothing but the variance $V(j)$ of $\ln c_{j,k} = \sum \ln W_{\epsilon_i}$, where (j,k) is the deepest common ancestor to the nodes (j_1,k_1) and (j_2,k_2) on the dyadic tree. This "ultrametric" structure of the correlation function shows that such a process is not stationary (nor ergodic) [14(c)]. However, we will generally consider uncorrelated consecutive realizations of length L of the same cascade process, so that, in good approximation, C depends only on the space lag $\Delta x = x_2 - x_1$, and one can replace ensemble average by space average. In that case, $C(\Delta k, j_1, j_2) = \langle C(k_1, k_1 + \Delta k, j_1, j_2)\rangle$ can be expressed as

$$C(\Delta k, j_1, j_2) = 2^{-(j-n)}\sum_{p=1}^{j-n} 2^{j-n-p}V(j-n-p), \tag{8}$$

where $j = \sup(j_1,j_2)$ and $n = \log_2\Delta k$.

Let us illustrate these features on some simple cases. For example, if one chooses, as in classical cascades, independent identically distributed (i.i.d.) random variables $\ln W_{\epsilon_i}$ of variance λ^2 (e.g., log normal), then $V(j) = \lambda^2 j$ and it can be established that, for $\sup(a_1,a_2) \le \Delta x < L$,

$$C(\Delta x, a_1, a_2) = \lambda^2\left[\log_2\left(\frac{L}{\Delta x}\right) - 2 + 2\frac{\Delta x}{L}\right]. \tag{9}$$

Thus, the correlation function decreases very slowly, independently of a_1 and a_2, as a logarithm function of Δx. This behavior is illustrated in Fig. 1 where a log-normal cascade has been constructed using Daubechies compactly supported wavelet basis (D-5) [16]. The correlation functions of the magnitudes of $f(x)$ have been computed as described above [Eq. (6)] using a simple box function for $\chi(x)$. Let us note that all the results reported in this Letter concern the increments of the considered signal. We have reproduced this analysis for wavelets of various shapes (e.g., successive derivatives of the Gaussian function) and checked that our results are actually independent of the specific choice of ψ. In Fig. 1(a) are plotted the "one-scale" ($a_1 = a_2 = a$) correlation functions for three different scales $a = 4$, 16, and 64. One can see that, for $\Delta x > a$, all the curves collapse to a single one, which is in perfect agreement with the expression (9): in semilog coordinates, the correlation functions decrease almost linearly (with slope λ^2) up to the integral scale L that is of order 2^{16} points. In Fig. 1(b) are displayed these correlation functions when the two scales a_1 and a_2 are different. One can check that, as expected, they still do not depend on the scales provided $\Delta x \ge \sup(a_1,a_2)$; moreover, they are again very well fitted by the above theoretical curve (except at very large Δx where finite size effects show up). The linear behavior of $C(\Delta x, a_1, a_2)$ vs $\ln(\Delta x)$ is characteristic for "classical" scale invariant cascades for which the random weights are uncorrelated. One can also consider not scale invariant cascades where these weights are not identically distributed and have an explicit scale dependence. For example, we can construct a log-normal model whose

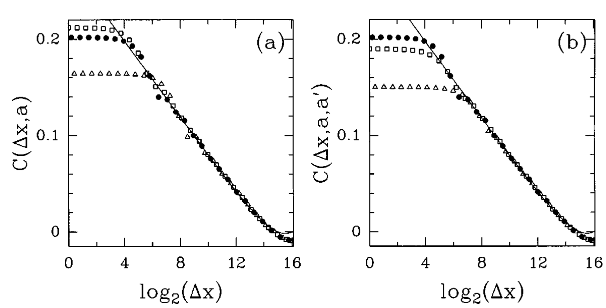

FIG. 1. Numerical computation of magnitude correlation functions for a scale invariant log-normal cascade process built on an orthonormal wavelet basis. (a) "One-scale" correlation functions $C(\Delta x, a, a)$ for $a = 4$ (□), 16 (●), and 64 (△). (b) "Two-scale" correlation functions $C(\Delta x, a, a')$ for $a = a' = 16$ (●), $a = 4$, $a' = 16$ (□), and $a = 16$, $a' = 64$ (△). The solid lines represent fits of the data with the log-normal prediction [Eq. (9)] using the parameters $\lambda^2 = 0.03$ and $\log_2 L = 16$.

coefficients $\ln c_{j,k}$ have a variance that depends on j as $V(j) = \lambda^2(2^{j\beta} - 1)/\beta \ln 2$. This model is inspired from the ideas of Castaing *et al.* [13] in their statistical study of velocity fluctuations in turbulence and reduces to a scale invariant model in the limit $\beta \rightarrow 0$ [19]. In this case, for $\sup(a_1, a_2) \leq \Delta x < L$, the correlation function becomes

$$C(\Delta x, a_1, a_2) = \frac{\lambda^2}{\beta \ln 2}\left[\frac{(\frac{L}{\Delta x})^\beta - \frac{\Delta x}{L}}{2^{\beta+1} - 1} - 1 + \frac{\Delta x}{L}\right]. \tag{10}$$

As for the first example, we have tested our formalism on this model constructed using the same Daubechies wavelet basis and considering, for the sake of simplicity, i.i.d. log-normal weights W_{ϵ_i}. Figures 2(a) and 2(b) are the analogs of Figs. 1(a) and 1(b). One can see that, when scale invariance is broken, our estimates of the magnitude correlation functions are in perfect agreement with Eq. (10) that predicts a power-law decrease of the correlation functions vs Δx.

The two previous examples illustrate the fact that magnitudes in random cascades are correlated over very long distances. Moreover, the slow decay of the correlation functions is independent of scales for large enough space lags ($\Delta x > a$). This is reminiscent of the multiplicative structure along a space-scale tree. These features are not observed in "additive" models like fractional Brownian motions whose long-range correlations originate from the sign of their variations rather than from the amplitudes. In Fig. 3 are plotted the correlation functions of an uncorrelated log-normal model constructed using the same parameters as in the first example, but without any multiplicative structure (the coefficients $c_{j,k}$ have, at each scale j, the same log-normal law as before but are independent) and for a fractional Brownian motion with $H = 1/3$. Let us note that from the point of view of both the multifractal formalism and the increment pdf scale properties, the "uncorrelated" and "multiplicative" log-normal models are undistinguishable since their one-point statistics at a given scale are identical. As far as the magnitude space-scale correlations are concerned, the difference between the cascade and the other models is striking: for $\Delta x > a$, the magnitudes of the fractional Brownian motion and of the log-normal "white noise" model are found to be uncorrelated.

The cascade concept is at the heart of a lot of phenomenological studies of turbulent flows. As a first application of our method, it is thus interesting to study magnitude correlations in fully developed turbulence data. Our experimental signals were recorded at the ONERA wind tunnel in Modane by Gagne and collaborators and represent the temporal fluctuations of the longitudinal velocity component. The Taylor scale based Reynolds number is $R_\lambda \simeq 2000$ and we processed about 2.5×10^8 points, i.e., about 25×10^3 integral scales, with a resolution of about twice the dissipative scale η. In all our computations, we use Taylor's hypothesis to identify temporal and spatial fluctuations. In Figs. 4(a) and 4(b) are plotted the "one-scale" and "two-scale" correlation functions. Both figures clearly show that space-scale magnitudes are strongly correlated. Very much like previous toy cascades, it seems that for $\Delta x > a$, all the experimental points $C(\Delta x, a_1, a_2)$ fall onto a single curve. We find that this curve is nicely fitted by Eq. (10) with $\beta = 0.3$, $\lambda^2 = 0.27$, and $L \simeq 2^{14}$ points. This latter length scale corresponds to the integral scale of the experiment that can be estimated from the power spectrum. It thus seems that the space-scale correlations in the magnitude of the velocity field are in very good agreement with a cascade model that is not scale invariant. This has been already observed by Castaing *et al.* [13] and further confirmed by other works [19] from "one-point" statistical studies. However, we have observed several additional features that do not appear in wavelet cascades. (i) For $\Delta x > L$, the correlation coefficient is not in the noise level ($C = 0$ as expected for uncorrelated events) but remains negative up to a distance of about 3 integral scales. This observation can be interpreted as an anticorrelation between successive eddies: very intense ones are followed by weaker ones, and conversely. (ii) For $\Delta x \simeq a$, there is a crossover from the

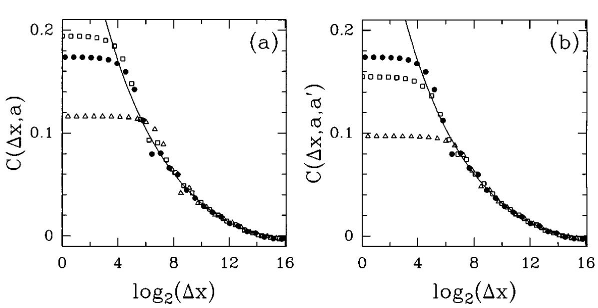

FIG. 2. Numerical computation of the magnitude correlation functions for a not scale invariant log-normal cascade process (see text). (a) "One-scale" correlation functions. (b) "Two-scale" correlation functions. Symbols have the same meaning as in Fig. 1. The solid lines correspond to Eq. (10) with $\beta = 0.3$, $\lambda^2 = 0.2$, and $\log_2 L = 16$.

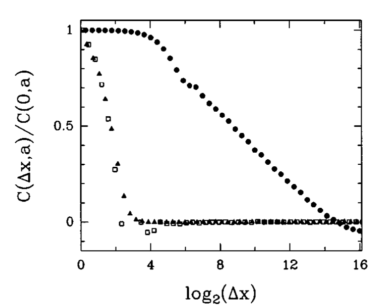

FIG. 3. "One-scale" ($a = 4$) magnitude correlation functions: log-normal cascade process (●), log-normal "white noise" (□), and $H = 1/3$ fractional Brownian motion (▲). Magnitudes are correlated over very long distances for the cascade process while they are uncorrelated when $\Delta x > a$ for the two other processes.

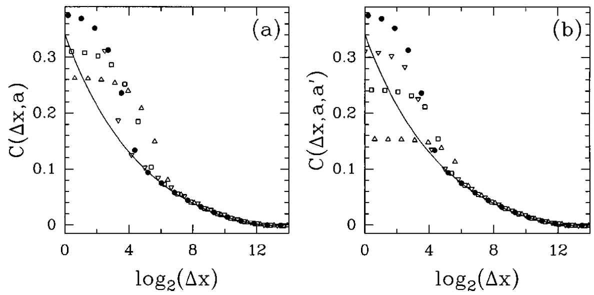

FIG. 4. Magnitude correlation functions of Modane fully developed turbulence data. (a) "One-scale" correlation functions at scales $a = 8$ (▽), 16 (●), 32 (□), and 64 (△). (b) "Two-scale" correlations functions at scales $a = 8$, $a' = 16$ (▽), $a = 16$, $a' = 16$ (●), $a = 16$, $a' = 32$ (□) and $a = 16$, $a' = 64$ (△). The solid lines correspond to a fit using Eq. (10) with $\beta = 0.3$, $\lambda^2 = 0.27$, and $\log_2 L = 13.6$. a, a', and Δx are expressed in experimental mesh size ($\simeq 2\eta$) units.

value $C(\Delta x = 0, a, a)$ (which is nothing but the variance of ω at scale a) down to the fitted curve corresponding to the cascade model. This was not the case in previous cascade models (Fig. 2). This observation suggests that simple self-similar (even not scale invariant) cascades are not sufficient to account for the space-scale structure of the velocity field. The interpretation of this feature in terms of correlations between weights at a given cascade step or in terms of a more complex geometry of the tree underlying the energy cascade is in progress.

To summarize, we have introduced a method that allows us to study the space-scale correlations in the magnitudes of random fractal functions. This method goes a step beyond classical one-point statistical analysis like pdf studies or multifractal formalism. It has been successfully tested on various toy cascade models. In turbulence we have been able to reveal that, if there is a cascade structure (with no correlation between weights at a given cascade step) that extends over the inertial range, it is definitely not scale invariant. Moreover, this process turns out to be less trivial than the simple picture provided by a Markovian multiplicative structure on a regular tree. Application of this method to other experimental situations looks very promising [21].

We are very grateful to Y. Gagne and Y. Malecot for providing their experimental signals. We acknowledge financial support from DRET under Contract No. 95/111.

[1] B. B. Mandelbrot, *The Fractal Geometry of Nature* (Freeman, San Francisco, 1982).

[2] U. Frisch and G. Parisi, in *Turbulence and Predictability in Geophysical Fluid Dynamics and Climate Dynamics*, edited by M. Ghil, R. Benzi, and G. Parisi (North-Holland, Amsterdam, 1985), p. 84.

[3] T. C. Halsey *et al.*, Phys. Rev. A **33**, 1141 (1986).

[4] G. Paladin and A. Vulpiani, Phys. Rep. **156**, 147 (1987).

[5] (a) T. Vicsek, *Fractal Growth Phenomena* (World Scientific, Singapore, 1989); (b) A. L. Barabási and H. E. Stanley, *Fractal Concepts in Surface Growth* (Cambridge University Press, Cambridge, 1995).

[6] C. Meneveau and K. R. Sreenivasan, J. Fluid Mech. **224**, 429 (1991).

[7] H. G. E. Hentschel, Phys. Rev. E **50**, 243 (1994).

[8] A. S. Monin and A. M. Yaglom, *Statistical Fluid Mechanics* (MIT Press, Cambridge, MA, 1975), Vol. 2.

[9] (a) A. N. Kolmogorov, C.R. Acad. Sci. USSR **30**, 301 (1941); J. Fluid Mech. **13**, 82 (1962). (b) A. M. Obukhov, J. Fluid Mech. **13**, 77 (1962).

[10] B. B. Mandelbrot, J. Fluid Mech. **62**, 331 (1974).

[11] E. A. Novikov, Phys. Fluids A **2**, 814 (1990); Phys. Rev. E **50**, 3303 (1994).

[12] U. Frisch, *Turbulence* (Cambridge University Press, Cambridge, England, 1995).

[13] (a) B. Castaing, Y. Gagne, and E. J. Hopfinger, Physica (Amsterdam) **46D**, 177 (1990); (b) B. Chabaud *et al.*, Phys. Rev. Lett. **73**, 3227 (1994); (c) B. Castaing and B. Dubrulle, J. Phys. II (France) **5**, 895 (1995).

[14] (a) M. E. Cates and J. M. Deutsch, Phys. Rev. A **35**, 4907 (1987); (b) A. P. Siebesma, in *Universality in Condensed Matter*, edited by R. Julien, L. Peliti, R. Rammal, and N. Boccara (Springer-Verlag, Heidelberg, 1988), p. 188; (c) J. O'Neil and C. Meneveau, Phys. Fluids A **5**, 158 (1993).

[15] (a) J. F. Muzy, E. Bacry, and A. Arneodo, Int. J. Bifurcation Chaos Appl. Sci. Eng. **4**, 245 (1994); (b) A. Arneodo, E. Bacry, and J. F. Muzy, Physica (Amsterdam) **213A**, 232 (1995).

[16] (a) Y. Meyer, *Ondelettes* (Hermann, Paris, 1990); (b) I. Daubechies, *Ten Lectures on Wavelets* (S.I.A.M., Philadelphia, 1992).

[17] F. Moret-Bailly *et al.*, C.R. Acad. Sci. Ser. 2 **313**, 591 (1991).

[18] (a) R. Benzi *et al.*, Physica (Amsterdam) **65D**, 352 (1993); (b) A. Arneodo *et al.*, J. Fourier Anal. Appl. (to be published).

[19] A. Arneodo, J. F. Muzy, and S. G. Roux, J. Phys. II (France) **7**, 363 (1997).

[20] Note that we do not exclude correlations along the branches of the tree.

[21] A. Arneodo, J. F. Muzy, and D. Sornette, "Causal Cascade in the Stock Market from the Infrared to the Ultraviolet" (to be published).