Theorem for Nonrotating Singularity-Free Universes

A. K. Raychaudhuri
Relativity and Cosmology Research Center, Physics Department, Jadavpur University, Calcutta 700032, India
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It is shown that all scalars built from the stress-energy tensor must have vanishing space-time average values in any nonrotating singularity-free universe in which the strong energy condition is satisfied. Application to the real universe, where observations seem to rule out such an "empty" universe, suggests that the hope of a reasonable realistic singularity-free cosmological model has to be abandoned. [S0031-9007(97)05165-X]

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The singularity theorems [1] of Hawking and Penrose led to a widely held belief that a time or null geodetic incompleteness is an essential feature of all relativistic cosmological solutions. That the proof of the theorem rested on a number of conditions was often overlooked. There were four notable conditions: (1) The causality condition requiring the nonexistence of closed timelike lines, (2) the strong energy condition $[T_{ik} - \frac{1}{2}Tg_{ik}]v^i v^k \geq 0$, (3) a generality condition on the Riemann-Christoffel tensor, and (4) the existence of a trapped surface. About the last condition, Misner, Thorne, and Wheeler [2] remark, "All the conditions except the trapped surface seem eminently reasonable for any physically realistic space time." It is interesting that the recently discovered singularity free solutions of Senovilla *et al.* [3,4] violate precisely the trapped surface condition, whereas the other three conditions hold good. True it is difficult to reconcile the Senovilla solutions with the characteristics of the presently observed universe; nevertheless, hopes have been raised that there may exist singularity free solutions which can serve as faithful models of the observed universe.

A look at the simplest Senovilla solution [3] reveals some interesting features. The space time is open in all the four dimensions but the physical and kinematic scalars all vanish so rapidly at spatial and temporal infinity that their space time averages taken over the entire space time vanish. The average of a quantity χ over the entire space time is defined as follows:

$$\langle\chi\rangle \equiv \left[\frac{\int_{-x_0}^{+x_0}\int_{-x_1}^{+x_1}\int_{-x_2}^{+x_2}\int_{-x_3}^{+x_3}\chi\sqrt{|g|}\,d^4x}{\int_{-x_0}^{+x_0}\int_{-x_1}^{+x_1}\int_{-x_2}^{+x_2}\int_{-x_3}^{+x_3}\sqrt{|g|}\,d^4x}\right]_{\lim x_o,x_1,x_2,x_3\to\infty}. \tag{1}$$

Of course, the average is defined only if the limit exists. For the Senovilla solution,

$$3\langle p\rangle = \langle\rho\rangle = 0\,,$$
$$\langle\theta^2\rangle = 0\,,$$
$$\langle\dot\theta\rangle = 0; \quad \langle\dot v^{\mu}_{;\mu}\rangle = 0\,.$$

All the above scalars appear linearly in the Raychaudhuri equation, and we present below a proof of the following general theorem:

In any singularity free nonrotating universe, open in all directions, the space-time average of all stress energy invariants including the energy density vanishes.

In the above, nonrotating means that the world lines of the matter in the universe form a normal (i.e., hypersurface orthogonal) congruence. The singularity free nature requires, in particular, the scalars from the Riemann tensor to have bounded values, and openness in all directions means that the space time has topology $R^3 \times R$.

For such a universe taking the x^0 axes along the world lines of matter, the metric may be written in the form

$$ds^2 = g_{00}dx^{0^2} + g_{\alpha\beta}dx^\alpha dx^\beta, \tag{2}$$

where the Greek indices run from 1 to 3. The domain of all the coordinates is from $-\infty$ to $+\infty$.

Our assumption about the openness in all directions means that the ratio of the volume of any three dimensional subspace to that of the entire space time vanishes, i.e.,

$$\frac{\int\int\int\sqrt{|^3g|}\,dx^i\,dx^k\,dx^l}{\int\int\int\int\sqrt{|g|}\,d^4x} = 0\,, \tag{3}$$

where the indices i,k,l are unequal and may refer to space or time coordinate, $|^3g|$ is the appropriate coefficient to give the invariant volume for the three dimensional element. For the unit vector v^i along the timelike coordinate x^0, we have the Raychaudhuri equation

$$\theta_{;i}v^i + \dot v^i_{;i} + \tfrac{1}{3}\theta^2 + 2\sigma^2 + k[T_{ik} - \tfrac{1}{2}g_{ik}T]v^i v^k = 0\,. \tag{4}$$

Taking the space time average of each term in the above equation, we get [here the space time averages are over infinite space time in the sense defined in (1)],

$$-\langle\dot v^i_{;i}\rangle - \langle\dot\theta\rangle = \tfrac{1}{3}\langle\theta^2\rangle + 2\langle\sigma^2\rangle + k\langle[T_{ik} - \tfrac{1}{2}g_{ik}T]v^i v^k\rangle\,. \tag{5}$$

With the strong energy condition $[T_{ik} - \frac{1}{2}g_{ik}T]v^i v^k \geq 0$ all the terms of the right-hand side are positive definite. Hence to get a positive value of the average density, we

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must have the left-hand side positive. The first term on the left gives

$$\langle \dot{v}^i_{;i} \rangle \equiv \frac{\int \dot{v}^i_{;i} \sqrt{-g}\, d^4x}{\int \sqrt{-g}\, d^4x} = 0\,.$$

The integral in the numerator can be converted to an integral of $\dot{v}^i$ over the three surface orthogonal to $\dot{v}^i$ at infinity. As $\dot{v}^i$ is orthogonal to v^i, this three surface contains two spacelike and one timelike dimension. In any case it is given by $\int |\dot{v}^i|\,|d\Sigma|$ where $|v^i|$ is the norm of the vector $\dot{v}^i$ and $|d\Sigma|$ is the proper volume of the orthogonal three dimensional element.

The velocity vector of matter v^i appears in the expression for T_{ik}. Thus the equation

$$R^i_k = -k[T^i_k - \tfrac{1}{2} T \delta^i_k]$$

makes v^i expressible as an algebraic expression of Ricci tensor components. In particular, if the matter is perfect fluid, v^i is the unit timelike eigenvector of R_{ik}. Hence, quite generally the kinematic variables like the acceleration $\dot{v}^i$, expansion θ will be determined by the Ricci tensor and its covariant derivatives. So an unbounded value of any kinematic scalar would mean scalars of the Riemann tensor blowing up and thus signal a singularity. We can hence take $|\dot{v}^i \dot{v}_i|, \theta, \dot{\theta}$, etc., to be bounded everywhere.

Consequently,

$$\langle \dot{v}^i_{;i} \rangle = \frac{\int |\dot{v}^i|\,|d\Sigma_i|}{\int \sqrt{-g}\, d^4x} = 0\,, \qquad (6)$$

where we have used (3).

In evaluating the value of $\langle \dot{\theta} \rangle$ defined by

$$\langle \dot{\theta} \rangle = \frac{\int \dot{\theta} \sqrt{-g}\, d^4x}{\int \sqrt{-g}\, d^4x} = 0\,,$$

we note that as $x^0 \longrightarrow \pm\infty$, $\dot{\theta}$ vanishes sufficiently rapidly so that θ remains finite for $x^0 \longrightarrow \pm\infty$. If $\sqrt{-g}$ is finite as $x^0 \longrightarrow \pm\infty$, then obviously the integral over x^0 in the numerator will converge to a finite value and consequently $\langle \dot{\theta} \rangle$ will vanish. If, however, $\sqrt{-g}$ blows up, the vanishing of $\dot{\theta}$ as $x^0 \longrightarrow \pm\infty$ may not make the x^0 integral convergent. Nevertheless, the vanishing of $\dot{\theta}$ would reduce the order of divergence of the numerator integral compared to the integral in the denominator and hence one again has $\langle \dot{\theta} \rangle = 0$. Consequently all the averages occurring in Eq. (5) vanish.

The generality of our treatment needs to be emphasized. The solutions of Senovilla type were based on the existence of doubtful symmetries and an apparently *ad hoc* splitting of metric tensor components into factors involving separately the time and space coordinates. Our result is based solely on the existence of a global time coordinate which is hypersurface orthogonal—one is tempted to identify it with the absence of rotation in the universe. The implicit idea in our discussion is that the gravitational collapse is arrested by the action of acceleration and that again means the existence of a nongravitational force. In such situations, our theorem shows that one has to sacrifice the idea of a finite average density. As one feels that observations rule out such an “empty” universe, the hope of a reasonable realistic singularity free cosmological solution has to be given up.

The question that naturally arises is the relation between the present theorem and the trapped surface condition. We have not addressed ourselves to this in the present discussion. Our condition seems physically more transparent.

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[1] S.W. Hawking and G.F.R. Ellis, *The Large Scale Structure of the Universe* (Cambridge University Press, Cambridge, 1973), Chap. 8, pp. 261–275.
[2] C.W. Misner, K.S. Thorne, and J.A. Wheeler, *Gravitation* (W.H. Freeman and Company, San Francisco, 1973), p. 935.
[3] J.M.M. Senovilla, Phys. Rev. Lett. **64**, 2219 (1990).
[4] E. Ruiz and J.M.M. Senovilla, Phys. Rev. D **45**, 1995 (1992).