High-Dimensional Bak-Sneppen Model

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We report on extensive numerical simulations on the Bak-Sneppen model in high dimensions. We uncover a very rich behavior as a function of dimensionality. For $d > 2$ the avalanche cluster becomes fractal and for $d \geq 4$ the process becomes transient. Finally, the exponents reach their mean field values for $d = d_c = 8$, which is then the upper critical dimension of the Bak-Sneppen model. [S0031-9007(98)06356-X]

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The Bak-Sneppen (BS) model [1], since its introduction, has attracted much attention in statistical physics. Thanks to its simplicity, it has led to a much deeper understanding of the nature of self-organized criticality [2] and of extremal dynamics [3] in general. The rules of the BS dynamics, in d dimensions, are very simple: the state of the model is completely defined by L^d numbers f_i arranged on a d-dimensional lattice of edge size L. At every time step the smallest of these numbers and its $2d$ nearest neighbors are replaced with new uncorrelated random numbers, drawn from the uniform distribution. Such very simple dynamics, based on the selection of the global minimum, is generally called extremal dynamics. It was first introduced in invasion percolation [4], and it results in a remarkably rich and interesting critical behavior.

The self-organized critical nature of the BS model (as well as of other extremal models) is revealed in its ability to naturally evolve towards a critical state where almost all of the variables f_i are above a threshold f_c. The dynamics in this state is characterized by scale-free bursts of activity or *avalanches*, which form a hierarchical structure [1,5] of subavalanches within bigger avalanches. This critical state is described by critical exponents. The distribution of avalanche duration s behaves as a power law $P(s) \sim s^{-\tau}$ with exponent τ. An avalanche of duration s covers a number of sites $V(s) \sim s^{\mu}$. The set of avalanche sites is generally fractal $V(s) \sim R(s)^{D_f}$, where $R(s)$ is the gyration radius and D_f is the (spatial) fractal dimension. The active site (the one with the global minimum f_i) has a dynamical wandering that can be described in terms of return times: The distribution $P_f(t) \sim t^{-\tau_f}$ of first return times is characterized by an exponent τ_f, whereas the distribution of all return times $P_a(t) \sim t^{-\tau_a}$ defines the exponent τ_a. For a random walk these two exponents take the values $\tau_f = 3/2$ and $\tau_a = 1/2$ in $d = 1$, whereas for $d \geq 2$ one finds $\tau_f = \tau_a = d/2$. Note that, with respect to previous literature [1,2], our notation is slightly different: The exponent μ in Ref. [2] is defined as $\mu = d/D$, where D was called "fractal dimension." Here we regard μ as a fundamental exponent (not a composite one) and reserve the name fractal dimension to describe the *spatial* geometrical properties of the avalanches. This choice, as we shall see, avoids ambiguities which can arise in higher dimensions, where the avalanche cluster becomes fractal.

A scaling theory was proposed in [2] which shows that scaling relations allow one to reduce the number of critical exponents to two, for example, μ and τ. Numerical simulations in $d = 1$ and 2 fully confirm the validity of the scaling theory [2]. The mean field limit, formally corresponding to the limit $d \to \infty$, has also been solved exactly [6]: the exponents, in this limit, take the values $\tau = \tau_f = \tau_a = 3/2$, $\mu = 1$, and $D_f = 4$. Finally, it was recently shown [7,8] that a further nontrivial relation, of a different nature, exists between μ and τ. This suggests that the BS universality class is characterized by a single exponent, e.g., $\mu(d)$, as a function of d.

In this Letter we analyze the behavior of the BS model as a function of dimensionality. The understanding of the critical behavior of a model, as a function of dimensionality, is a central issue in statistical physics. In particular, the identification of the upper critical dimension d_c, above which the mean field picture applies, is of great importance. Indeed, it allows one to understand the behavior of a finite dimensional system using the powerful tools of dimensional (ϵ) expansion. In equilibrium statistical mechanics, this is almost routine work, but for nonequilibrium systems it is still a challenging issue. For this reason the understanding of the behavior of simple models as a function of dimensionality is of great importance.

We present extensive numerical simulations for the BS model which show that (i) the upper critical dimension is $d_c = 8$, (ii) for $d > 2$ the avalanches are no more compact $D_f < d$, (iii) in $d = 3$ we find $\mu < \tau_a < 1$, and (iv) for $d \geq 4$ the process becomes transient, i.e., $\tau_a = \tau_f > 1$. The BS model then shows a quite rich behavior, with four qualitatively different regimes ($d \leq 2$, $2 < d < 4$, $4 \leq d < 8$, and $d \geq 8$), as a function of dimensionality. In particular, we find that the relation $\mu = \tau_a$ holds up to only $d = 2$ (note that this relation

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is in any case problematic close to the mean field limit $\mu = 1, \tau_a = 3/2$). A possible interpretation of our results is that "geometric" exponents, such as τ_a, τ_f, and D_f become independent from "avalanche" or "memory" (see later) exponents for $d > 2$ and the number of independent exponents changes with dimension. Indeed, since $D_f < d$ and $\tau_a > \mu$, for $d > 2$, we find that the BS critical behavior is determined by three independent critical exponents. We shall first discuss in detail the numerical procedure, defining operationally the quantities we measure. Then we present the numerical results, and finally we discuss them.

The BS dynamics can be simulated very efficiently using a treelike search and replace algorithm [9]. This decreases greatly computation times, so that memory is the only limitation to the system sizes studied. The numerical procedure was first tested in $d = 1$ and $d = 2$, and we found complete agreement with Refs. [2,9]. In order to compute the exponents μ and τ, following Ref. [10], we introduce an *age* variable k_i on each site. The age k_i of site i measures the time elapsed since the last update of the variable f_i. This method enables us to give a precise evaluation of the exponent μ. Indeed at each time the sites with age k_i less than s identify the current avalanche of duration s. Therefore $V(s)$ is simply obtained counting the sites with $k_i < s$. Age variables also allow for a determination of the avalanche exponent τ. In systems evolving by an extremal dynamics, it has been found [3,10] that the probability that the global minimum f_i occurs on a site with age k_i behaves as $\rho(k_i) \sim k_i^{-\alpha}$. The exponent $\alpha > 0$ implies that older sites are less likely to be selected. This reflects the fact that a very old site has survived many selections, and therefore it has very likely a high value of the variable f_i: The more it survives the higher is its f_i, possibly even greater than the threshold f_c, in which case the site will never be selected before a new update (occurring if one of its neighbors is selected). The exponent α is related to the avalanche exponent by $\alpha = 3 - \tau$ [11]. The advantage of measuring α, with respect to a direct measure of τ, is that the statistics of the former is much richer than that of the latter in the same simulation. We measured τ in both ways and found good agreement (which also supports the validity of the relation $\alpha = 3 - \tau$). Since statistical uncertainty of the exponent α is much less than that of τ, we report here only the value $\tau = 3 - \alpha$. The return times exponents are measured in the usual way [2]: let $t_i^{(k)}$, $k = 0, 1, \ldots$ be the (return) times when site i is visited ($t_i^{(k)} < t_i^{(k+1)}$). The first return exponent is obtained from the statistics of $t_i^{(k)} - t_i^{(k-1)}$, whereas the all returns exponent is obtained from $t_i^{(k)} - t_i^{(0)}$. Finally, in order to obtain the fractal dimension D_f, we compute the gyration radius of avalanches of size $V(s)$.

The numerical results are summarized in Table I. As shown in Fig. 1, our numerical results for μ and τ are in good agreement with the exact relation recently found in Ref. [8]. This is an important consistency check for the reliability of the simulations. In $d = 2$ we find a slight difference between τ_a and μ which is, however, within error bars. The data are also plotted in Fig. 2 for completeness. For $d = 3$ the difference between τ_a and μ is much larger than the error bars (see Fig. 2). In Fig. 3 we plot the exponents as a function of dimensionality. For $d \leq 3$ the process is recurrent, since $\tau_a < 1$, i.e., each visited site, in an infinite system, is visited again an infinite number of times (or, stated differently, a selected site will be selected again with probability one). Instead for $d \geq 4$ the process becomes transient, i.e., $\tau_a = \tau_f > 1$: each site, in an infinite system, is visited a finite number of times (there is a finite, smaller than one, probability that a selected site will be selected again). The relation $\tau_a + \tau_f = 2$ if $\tau_a < 1$ and $\tau_a = \tau_f$ if $\tau_a \geq 1$ (a classical result from renewal theory [12]) is always respected, the former holding for $d \leq 3$. The return time statistics, in the BS model, is determined both by memory effects and by geometry. Activity can return to

TABLE I. Exponents of the BS model for different dimensions d. An upper bound on the error is ± 0.01 for safety, even if for some of the listed exponents confidence is greater. In particular, for $d = 7$, the values are distinguishable from the $d = 8$ ones. The last line gives the size $L = 2^n$ of the edge of the hypercube used for the simulations.

d	1	2	3	4	5	6	7	8
μ	0.42	0.69	0.85	0.92	0.95	0.96	0.98	1.00
τ	1.07	1.25	1.35	1.41	1.45	1.46	1.48	1.50
τ_a	0.42	0.70	0.92	1.15	1.29	1.40	1.49	1.50
τ_f	1.58	1.28	1.09	1.16	1.29	1.40	1.49	1.50
D_f	1	2	2.6	3.3	...	...	...	...
n	21	10	7	5	4	3	3	2

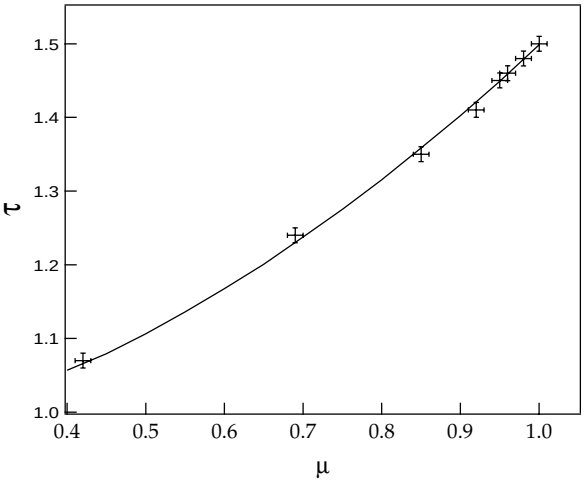

FIG. 1. Avalanche exponent τ vs μ for different dimensions (crosses). The continuous line is the exact relation between the two exponents as from Ref. [7]. The data from the simulations and the exact relation are in excellent agreement.

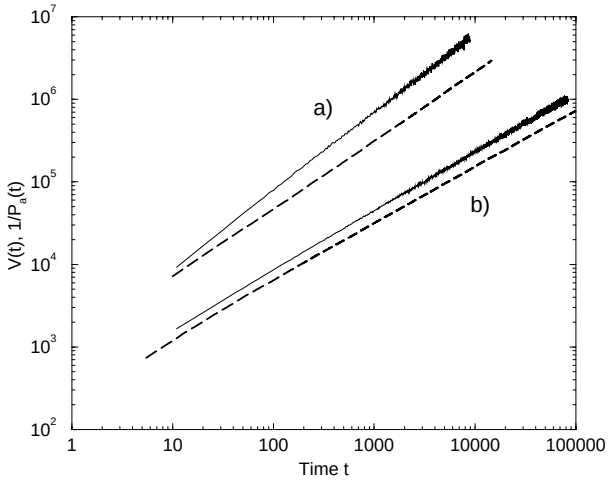

FIG. 2. Log-log plot of the number of covered sites $V(t)$ (dashed line) and the (inverse) all return times distribution $P_a(t)$ (full line) as a function of time t. The results in (a) $d = 3$ show that the slope of the two quantities are clearly different (hence $\mu \neq \tau_a$); (b) $d = 2$ results show instead full compatibility of the slopes with $\mu = \tau_a$.

site i either because all sites with variables larger than f_i are eliminated by the BS dynamics (which we call a memory return) or because the activity returns close to site i and $f_i \rightarrow f_i'$ is updated (geometric return). As the dimension d increases, geometric returns become less and less relevant (see later), and for very large dimensions one expects to recover the mean field result $\tau_a = \tau_f = 3/2$. We find that the mean field limit holds for $d \geq d_c = 8$.

We now turn to the discussion of the onset of fractality of avalanche clusters for $d > 2$ (the possibility of fractal avalanche clusters was already mentioned in [2], but not fully explored since their $d = 1, 2$ simulations yelded compact avalanches). At $d = 2$, we found numerically that the fractal dimension is slightly smaller than $D_f = 2$. We argue, however, that small size effects occur and that $D_f = 2$ holds. Indeed, as the system size L is increased, the slope of the curve $\log V(s)$ vs $\log R(s)$ increases, suggesting that avalanches are compact. The occurrence of small size effects can be understood analyzing the dynamics of the growth of the avalanche cluster in d dimensions. Indeed, the avalanche cluster growth in $d = 2$ is characterized by more and more returns of the activity to the bulk as time goes by. Therefore the process has many opportunities to fill any hole that was left behind and that was giving fractal features to the cluster. As a consequence, long avalanches are compact objects. Yet, at an early growth stage, the surface-to-bulk ratio is large ($\sim 2d$), returns to the bulk are rare, and growth takes place most likely on the cluster surface. In a standard model of growth, where new sites can be added only at the surface of the cluster, memory effects have been recently investigated in Ref. [13]. There it was shown that when the memory exponent $\alpha > 1$ the cluster has a fractal dimension $D_f < d$. Small avalanches can therefore show fractal features that disappear on longer times due to returns to the bulk. These small size effects are also visible in the distribution of first return times (see also [2]). In $d \geq 3$ the same size effect should appear; however, in this case we find a fractal dimension which is definitely smaller than d [14]. Returns to the bulk are fewer in $d \geq 3$ than in $d = 2$, and more importantly the topology of a fractal cluster is very different: while the two-dimensional cluster is characterized by a distribution of holes of all sizes, a fractal in $d \geq 3$ is most likely a ramified object. Returns to the bulk can fill the holes of the $d = 2$ cluster but it is much more difficult for them to turn the branched fractal structure of the $d \geq 3$ avalanche into a compact one. For this reason, we believe that our data are compatible with the occurrence of a compact cluster in $d = 2$ and with $D_f < d$ when $d \geq 3$. In any case, numerical data are relatively stable with changes of the system size, and we could not detect any small size effect such as the one discussed above for $d = 2$.

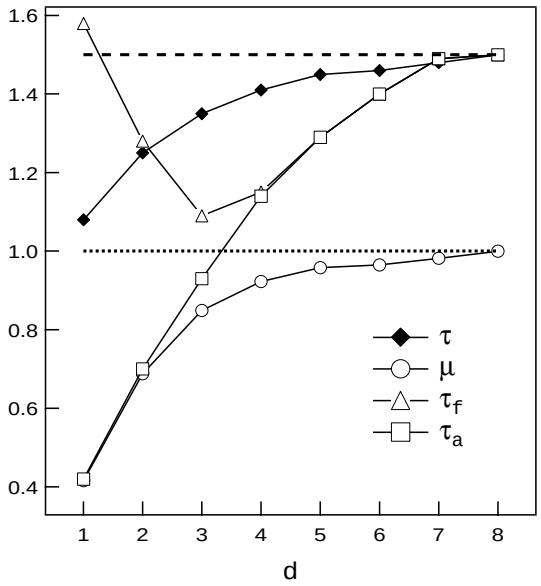

FIG. 3. The exponents μ, τ, τ_a, and τ_f as a function of dimensionality. Dashed lines at 1 and 1.5 have been drawn for reference.

Geometric returns arise because the avalanche cluster has many self-intersections: indeed, the self-intersection set has a fractal dimension $D_I = 2D_f - d > 0$ for $d < 8$ [15]. Note that D_I is smaller than D_f which means that the larger the avalanche the smaller is the fraction of intersection sites. Moreover, as mentioned earlier, geometric returns become less and less relevant as d increases since $D_f - D_I = d - D_f$ also increases with d. Above $d = d_c = 8$ the fractal dimension $D_f = 4$ attains its mean field value and the avalanche has no self-intersections ($D_I < 0$).

The jump probability distribution $\rho(r)$ was analyzed in Ref. [2] under the hypothesis of a compact cluster.

It is defined as the probability that the activity jumps in one time step to a site at a distance r from the current site and it falls off with a power law behavior: $\rho(r) \sim r^{-\pi}$. For fractal avalanches, the exponent π was related to τ and D_f in Ref. [5] by $\pi = 1 + D_f(2 - \tau)$. It is interesting to observe that, for the values quoted in Table I, π is always slightly larger than 3 (its mean field value being exactly 3), so that the second moment of $\rho(r)$ is finite; an uncorrelated random walk with such a jump distribution would show the usual random walk behavior, $\tau_a = d/2$, and would become transient above $d = 2$. The differences with respect to the random walk behavior are therefore a strong indication of correlations induced by memory effects.

In order to check the full consistency of the results shown in Table I, we are also exploring an alternative way to move away from the mean field limit [16]. Within a $d = 1$ system, we choose the "nearest neighbors" of the active site at random over the lattice, with a probability which is a power law decreasing function of the distance from the active site, with exponent ω. Preliminary results show that the mean field limit is recovered when $\omega \to 1$. Varying $\omega > 1$ again three different critical regimes appear. In particular, we find a region of the values of ω where activity is recurrent ($\tau_a < 1$) but $\mu \neq \tau_a$ and a region where activity is transient ($\tau_a > 1$) but $\mu < 1$. Moreover we find that, whenever μ takes the same values listed in Table I, also the other quantities take on the same corresponding values (apart, of course, from D_f which is always smaller than 1). Finally, working in $d = 1$ allows us to simulate extremely large sizes and to rule out any finite size effects, thus strengthening the reliability of the results shown in Table I.

It has been argued that the mean field limit of BS can be described as a branching process [6]. The value $d_c = 8$, which coincides with the upper critical dimension of branched polymers, is consistent with this picture [17]. We believe that close to $d = 8$ branched polymers give a reasonable description of the geometry of the BS process. An $\varepsilon = d_c - d$ expansion for the BS model could therefore be feasible, also using the recent expansion around the mean field solution of Ref. [8]. In this respect, our results allow for a prediction of the first coefficient $\mu \simeq 1 - 0.017\varepsilon$ of the ε expansion.

In summary, we have presented numerical results for the BS model in high dimensions. These allow one to conclude that for $d > 2$ avalanches become fractal and for $d \geq 4$ the process becomes transient. Finally the mean field limit is reached at the upper critical dimension $d_c = 8$. This behavior, with three different nontrivial regimes of criticality, is substantially richer than that of equilibrium statistical models, where nontrivial behavior occurs in only one region of dimensionality, limited by the lower and the upper critical dimensions. Our results also call for a close analysis of the scaling theory of Ref. [2], which has been developed under the implicit assumptions of a recurrent process with compact avalanches.

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[14] The measurement of the fractal dimension is the most time consuming, and moreover the feasible system sizes are smaller the larger the dimension. For this reason we did not push the calculation of D_f to $d > 4$. Note that the mean field result $D_f = 4$ implies $D_f \leq 4$ and therefore a fractal structure for $d \geq 5$.

[15] The dimension D_I of the intersection of two objects of fractal dimension D_f in d dimensions can be obtained knowing that the codimension of the intersection ($d - D_I$) is the sum of the codimensions of the two intersecting objects ($d - D_f$). Therefore $d - D_I = 2(d - D_f)$ from which the relation $D_I = 2D_f - d$ follows.

[16] P. De Los Rios *et al.* (to be published).

[17] Indeed $d = 8$ is the upper critical dimension of branched polymers as obtained using simple Flory arguments, see, e.g., J. Isaacson and T. C. Lubensky, J. Phys. (Paris), Lett. **41**, 469 (1980), and references therein.