Nonperturbative Renormalization Group for Chaotic Coupled Map Lattices

Anaël Lemaître[1,2] and Hugues Chaté[2,1]
[1]LadHyX, Laboratoire d'Hydrodynamique, Ecole Polytechnique, 91128 Palaiseau, France
[2]CEA, Service de Physique de l'Etat Condensé, Centre d'Etudes de Saclay, 91191 Gif-sur-Yvette, France
(Received 26 January 1998)

A nonperturbative renormalization group is derived for chaotic coupled map lattices (CMLs) with diffusive coupling, leading to a natural space-continuous limit of these systems. We show that, under very general conditions, the universal properties of the local map are translated to the spatiotemporal level, demonstrating the self-similarity of the bifurcation diagrams of strongly coupled CMLs and the accompanying divergence of length scales. [S0031-9007(98)06460-6]

PACS numbers: 05.45.+b, 05.70.Ln, 64.60.Ak

The use of renormalization group (RG) ideas to unveil the universal features of the cascades of bifurcations of nonlinear dynamical systems with few degrees of freedom has played a major role in our understanding of (temporally) chaotic systems [1]. Simple maps of the real interval such as the logistic map have been the models of choice on which most theoretical advances were made [2–4]. No similarly general framework is available, however, for *spatiotemporal chaos* when the basic equations cannot be legitimately reduced to the interaction of a few modes. This holds even for simple models such as coupled map lattices (CMLs), i.e., discrete-time discrete-space dynamical systems in which maps, arranged at the nodes of a lattice, interact locally [5]. Literature dealing with the question of the extension of single-map RG to CMLs does exist [6], but it is restricted to perturbative treatments around the accumulation points of bifurcation cascades of the local map. Moreover, it usually considers the small coupling limit and small deviations from spatially homogeneous solutions in one or two space dimensions. One notable exception is the numerical work of van de Water and Bohr [7], who showed numerically that many quantities of interest do exhibit scaling properties related to those of the local map, even for rather large values of the coupling.

In this Letter, we introduce a nonperturbative renormalization group approach to CMLs with linear diffusive coupling which translates to the spatiotemporal level the universal features of the local maps involved. We show that, under broad conditions, the bifurcation diagrams of CMLs present the same self-similarity as that of their local maps, with the same coupling-independent accumulation point, around which length scales diverge with a universal scaling related to the diffusive coupling. Our work also leads to a definition of a "natural" continuous-space limit for CMLs which can be seen as a good starting point for analytical approaches of spatiotemporal chaos. We believe that these results are relevant to general reaction-diffusion systems. Our findings extend the scope of previous studies [6,7]. They are valid in chaotic regimes, are not restricted to the vicinity of accumulation points, and apply to large classes of solutions in all dimensions. They are demonstrated exactly for coupled tent maps, while they are valid asymptotically for more general local functions such as the logistic map.

For simplicity, we consider real variables X lying at the nodes of a d-dimensional hypercubic lattice $\mathcal{L}$. They are updated synchronously at discrete time steps by

$$\mathbf{X}^{t+1} = \mathbf{\Delta}_g \circ \mathbf{S}_\mu(\mathbf{X}^t), \tag{1}$$

where $\mathbf{X}^t = (\mathbf{X}^t_{\vec{r}})_{\vec{r}\in\mathcal{L}}$ represents the state of the lattice at time t, $\mathbf{S}_\mu$ transforms each variable by the local map S_μ, and $\mathbf{\Delta}_g$ is the diffusive coupling operator

$$[\mathbf{\Delta}_g(\mathbf{X})]_{\vec{r}} = (1 - 2dg)\mathbf{X}_{\vec{r}} + g \sum_{\vec{e}\in\mathcal{V}} \mathbf{X}_{\vec{r}+\vec{e}}\,, \tag{2}$$

with g the coupling strength and $\mathcal{V}$ the set of the $2d$ nearest neighbors of site $\vec{0}$. Without loss of generality, we consider the family of local maps

$$S_\mu(X) = 1 - \mu|X|^{1+\varepsilon} \quad \text{with} \quad \mu \in [0,2] \quad \text{and} \quad \varepsilon > 0\,, \tag{3}$$

which leave the $[-1,1]$ interval invariant and, in particular, the tent ($\varepsilon = 0$) and the logistic ($\varepsilon = 1$) maps.

We first present the collective behavior observed for strongly coupled, chaotic CMLs, and their self-similar bifurcation diagrams, and then define the strong coupling limit to which our study is especially relevant.

In the "chaotic region" of the local map ($\mu > \mu_\infty$ with $\mu_\infty = 1$ for the tent map, and $\mu_\infty = 1.401\ldots$ for the logistic map), and with the strong, "democratic," equal-weight coupling $g = 1/(2d + 1)$, the CMLs defined above are extensively chaotic (e.g., their Lyapunov dimension is proportional to their size). Almost all initial conditions flow to one of a few attractors which possess a well-defined infinite-size, infinite-time limit and can be characterized by the evolution of spatial averages. The corresponding dynamics has been termed nontrivial collective behavior (NTCB) [8] to emphasize the emergence of a macroscopic evolution in the presence of microscopic chaos. Figure 1 shows the bifurcation diagram

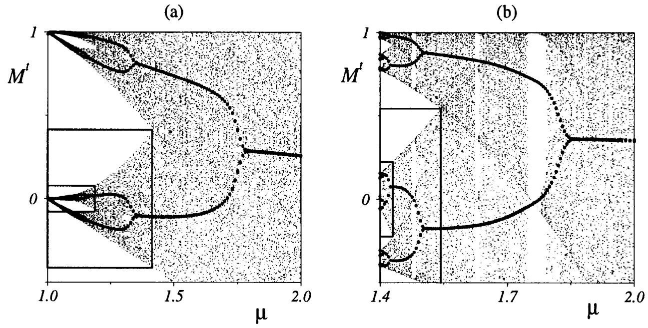

FIG. 1. Democratically coupled ($g = 0.2$) tent (a) and logistic (b) maps on a $d = 2$ lattice of linear size $L = 2048$ with periodic boundary conditions: Bifurcation diagram of $M^t = \langle X \rangle^t$ (filled circles) superimposed on that of the local map S_μ (small dots). The $[\mu_\infty, \bar{\mu}_1] \otimes I^1_{\bar{\mu}_1}$ and $[\mu_\infty, \bar{\mu}_2] \otimes I^2_{\bar{\mu}_2}$ regions are shown. For coupled tent maps, these regions transformed by the RG coincide with the bifurcation diagrams of $\mathbf{\Delta}^2_g \circ \mathbf{S}_\mu$ and $\mathbf{\Delta}^4_g \circ \mathbf{S}_\mu$, themselves indistinguishable from the whole figure. For the logistic maps, the agreement is poorer at such orders due to the inexactness of the RG for S_μ.

of the simplest spatial average, $M^t \equiv \langle X \rangle^t$, for $d = 2$ lattices of coupled tent and logistic maps. Decreasing μ at fixed g, periodic NTCB of period $1, 2, 4, 8, \ldots$ is observed. These collective period-doubling bifurcations are, in fact, Ising-like phase transitions [9]. The instantaneous distribution $p^t(X)$ of site values, smooth and well defined in the infinite-size limit, follows the same collective behavior. Only periodic collective motion has been observed for $d = 2$ and 3 (Fig. 2b), while more complex NTCB (e.g., quasiperiodic) exists for $d > 3$. The bifurcation diagram of M^t is reminiscent of the self-similar band structure of the local map, but the critical points μ^c_n of the phase transitions differ from the band-splitting points $\bar{\mu}_n$ of S_μ (Fig. 1). Approaching μ_∞, however, it rapidly becomes numerically impossible to resolve the particular NTCB observed, because prohibitively large lattices, as well as an increasing numerical resolution, are then required. If the periods observed are clearly rather large (at least eight in Fig. 1), the uniqueness of attractors near μ_∞ cannot be ascertained, and no evidence of an infinite cascade of phase transitions is available. The RG approach below clarifies these points.

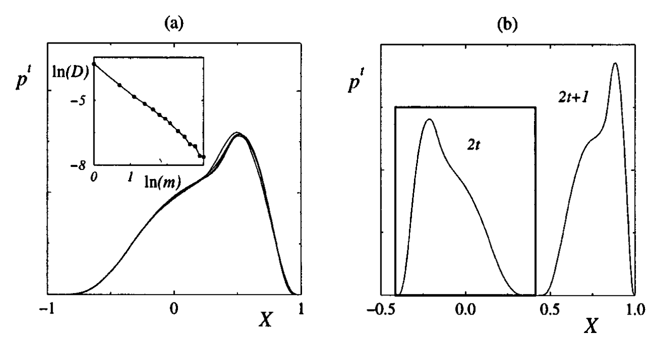

FIG. 2. Asymptotic (large t) single-site distributions p^t for democratically coupled tent maps ($d = 2$, $L = 2048$). (a) Stationary states at $\mu = 2$ for $1 \leq m \leq 32$. For $m > 1$, the distributions cannot be separated on this graph; they converge to a universal distribution of the continuous limit. Inset: log-log plot of the distance $D^2 = \int \mathrm{d}X\,(p_m - p_{32})^2$ vs m. (b) Period-2 collective cycle at $\mu = \bar{\mu}_1$ for $m = 1$; the distribution in the rectangle (even time steps) is transformed exactly by the RG onto that for $m = 2$ at $\mu = 2$ (a).

The NTCB described above is observed for $g = 1/(2d + 1)$. As a matter of fact, it can be observed for all g values beyond a well-defined threshold, as argued below. Consider a CML with a local map S_μ in a two-band chaos regime and the coarse-grained patterns formed by considering only in which band each site lies at large, even, time steps. For weak (and zero) coupling, there exist "frozen" patterns of this type corresponding to different chaotic attractors or ergodic components, each with a finite basin of attraction [10]. Recent work [11] shows that there exists, at a given μ, a limit value $g_e(\mu)$ beyond which no frozen pattern exists (excluding configurations whose basin of attraction is of the measure of zero in phase space). It is in this strong-coupling limit $g \geq g_e(\mu)$ that NTCB is observed. Generally, $g_e(\mu)$ decreases weakly with μ [11], but, again, it is difficult to estimate near μ_∞. On the other hand, it is convenient to define $g^*_e = \max_\mu[g_e(\mu)]$, and to consider values of g above g^*_e. For the $d = 2$ lattices of tent and logistic maps considered above, $g^*_e \simeq 0.10$ [11].

We now briefly review how RG is defined for single maps, before considering the case of CMLs. For clarity's sake, we use the tent map, for which the RG can be performed exactly. The invariant interval of S_μ is $I^0_\mu = [1 - \mu, 1]$ for all $\mu \leq \bar{\mu}_0 = 2$. The second iterate of the map, restricted to I^1_μ, the "central" band (containing $X = 0$) of the two-band region of S_μ is written

$$S^2_\mu|_{I^1_\mu} = h^{-1}_\mu \circ S_{q(\mu)}|_{I^0_{q(\mu)}} \circ h_\mu\,, \tag{4}$$

with $h_\mu(X) = X/(1 - \mu)$ and $q(\mu) = \mu^2$. Whenever $q(\mu) \leq \bar{\mu}_0$, this relation may be interpreted in terms of an equivalent variable $X_1 = h_\mu(X) \in I^0_{q(\mu)}$ governed by the map $S_{q(\mu)}$. Therefore, for all $\mu \leq \bar{\mu}_1 = q^{-1}(\bar{\mu}_0)$—here, $\bar{\mu}_1 = \sqrt{2}$—the invariant intervals of S^2_μ are $I^1_\mu = h^{-1}_\mu(I^0_{q(\mu)})$ and $I^{1'}_\mu = S_\mu(I^1_\mu)$. Relation (4) is easily generalized to any iterate of S_μ. It insures that S_μ displays a self-similar cascade of band-splitting points $\bar{\mu}_n$, below which regimes with more than 2^n bands are observed. When $n \to \infty$, $\bar{\mu}_n$ converges to μ_∞ and the intervals shrink to zero with the so-called Feigenbaum constants δ and α depending only on ε.

For more general maps ($\varepsilon > 0$), the RG equation (4) holds for some function q and some continuous bijection h_μ which cannot be derived exactly. Several approximations can be made. For example, expanding S^2_μ around $X = 0$ is the so-called one-parameter centered approximation of the RG. Higher-order, noncentered, multiparameter approximations are also possible [4]. They all provide estimates of the points $\bar{\mu}_n$, which converge when $n \to \infty$ toward an effective accumulation point with Feigenbaum exponents close to the actual ones.

Let us now apply the same ideas to the CML defined above, and derive an expression for the second iterate $(\mathbf{\Delta}_g \circ \mathbf{S}_\mu)^2$ of the evolution operator. Take $\mu \leq \bar{\mu}_1$ to insure that the local map has at least two bands. Let $\mathbf{I}_\mu^1$ ($\mathbf{I}_\mu^{1'}$) denote the set of lattice configurations for which all variables $\mathbf{X}_{\vec{r}} \in I_\mu^1$ ($I_\mu^{1'}$). Since $\mathbf{\Delta}_g$ keeps all such intervals invariant, $\mathbf{I}_\mu^1$ and $\mathbf{I}_\mu^{1'}$ are not only exchanged by the action of $\mathbf{S}_\mu$, but also by $\mathbf{\Delta}_g \circ \mathbf{S}_\mu$, and are thus invariant under $(\mathbf{\Delta}_g \circ \mathbf{S}_\mu)^2$: they constitute generalized bands in the phase space of the CML. For any $\mu \leq \mu_1$, the operator $(\mathbf{\Delta}_g \circ \mathbf{S}_\mu)^2$ restricted on the central band $\mathbf{I}_\mu^1$ can thus be written

$$(\mathbf{\Delta}_g \circ \mathbf{S}_\mu)^2|_{\mathbf{I}_\mu^1} = \mathbf{\Delta}_g \circ \mathbf{S}_\mu|_{\mathbf{I}_\mu^{1'}} \circ \mathbf{\Delta}_g \circ \mathbf{S}_\mu|_{\mathbf{I}_\mu^1}\,.$$

We now consider again tent maps, while the general case will be discussed later. In this case, $\mathbf{S}_\mu|_{\mathbf{I}_\mu^{1'}}$ is linear, since for all $X \in I_\mu^{1'}$, $S_\mu(X) = 1 - \mu X$. Therefore, it commutes with $\mathbf{\Delta}_g$ and we have

$$(\mathbf{\Delta}_g \circ \mathbf{S}_\mu)^2|_{\mathbf{I}_\mu^1} = \mathbf{\Delta}_g^2 \circ \mathbf{S}_\mu^2|_{\mathbf{I}_\mu^1}\,. \qquad (5)$$

Using the RG equation (4) and the linearity of the operator $\mathbf{h}_\mu$ induced by h_μ on lattice configurations, we write

$$(\mathbf{\Delta}_g \circ \mathbf{S}_\mu)^2|_{\mathbf{I}_\mu^1} = \mathbf{h}_\mu^{-1} \circ \mathbf{\Delta}_g^2 \circ \mathbf{S}_{q(\mu)}|_{\mathbf{I}_{q(\mu)}^0} \circ \mathbf{h}_\mu\,.$$

This equation involves a CML in which the coupling operator is applied twice. It is easily extended to generalized CML of the form $\mathbf{\Delta}_g^m \circ \mathbf{S}_\mu$ where the coupling step is applied m times, yielding

$$(\mathbf{\Delta}_g^m \circ \mathbf{S}_\mu)^2|_{\mathbf{I}_\mu^1} = \mathbf{h}_\mu^{-1} \circ \mathbf{\Delta}_g^{2m} \circ \mathbf{S}_{q(\mu)} \circ \mathbf{h}_\mu\,, \qquad (6)$$

which constitutes a RG equation for CMLs in the (m, μ) parameter space.

When $\mu \leq \bar{\mu}_n$, the local map has 2^n bands. If all the sites are, for example, taken inside I_μ^n, the central band of order n, then they lie at all times in one of the 2^n bands. In the following, we call this situation a *banded state of order n* [12]. We can thus write, generalizing (5)

$$(\mathbf{\Delta}_g^m \circ \mathbf{S}_\mu)^{2^n}|_{\mathbf{I}_\mu^n} = \mathbf{\Delta}_g^{2^n m} \circ \mathbf{S}_\mu^{2^n}|_{\mathbf{I}_\mu^n}\,. \qquad (7)$$

This allows one to generalize (6) to banded states of order n.

More general maps ($\varepsilon > 0$) are not linear on $I_\mu^{1'}$, but they are invertible on this interval (the critical point $X = 0$ is in the other band). They can be approximated by a nonvanishing tangent and the calculations above can be repeated. For states of increasing periodicity ($\mu \leq \bar{\mu}_n$), the restrictions of S_μ on each of its 2^n bands but the central band $I_\mu^n \ni 0$, are better and better approximated by tangents, since the diameters of these intervals shrink with increasing n. Finally, since h_μ is generally linear, Eq. (6) holds as an approximation of the RG for CMLs exact for coupled tent maps.

RG equation (6) has been derived for the restriction of the evolution operator to $\mathbf{I}_\mu^1$ and $\mathbf{I}_\mu^{1'}$ and for $\mu \leq \bar{\mu}_1$ (banded states of the order of 1). *A priori*, banded states are only some of the many possible chaotic attractors of our CML for $\mu > \mu_\infty$. Suppose, however, that $g \geq g_e^*$ so that we are in the NTCB regime. If, furthermore, $\mu \leq \bar{\mu}_1$, almost all initial conditions eventually belong to $\mathbf{I}_\mu^1$ [13], and Eq. (6) can be applied once: for any $\mu \leq \bar{\mu}_1$, the behavior of the original CML is equivalent to that of $\mathbf{\Delta}_g^2 \circ \mathbf{S}_{\mu'}$ for $\mu' = q(\mu) \leq 2$. Without studying the collective behavior of $\mathbf{\Delta}_g^2 \circ \mathbf{S}_{\mu'}$ itself, it is already clear that if $\mathbf{\Delta}_g \circ \mathbf{S}_{\mu'}$ brings all sites into the same band, a second application of $\mathbf{\Delta}_g$ at each time step must "synchronize" the sites even more: $\mathbf{\Delta}_g^2$ is a "stronger" coupling than $\mathbf{\Delta}_g$. All generalized CMLs with $m \geq 1$ are in the strong-coupling regime whenever $g \geq g_e^*$. We can write, symbolically: $g_e^*(m+1) \leq g_e^*(m)$. Therefore, since $\mu' \leq \bar{\mu}_1$ when $\mu \leq \bar{\mu}_2$, then $\mathbf{\Delta}_g^2 \circ \mathbf{S}_{\mu'}$ reaches a banded state of the order of 1, corresponding to a banded state of the order of 2 for $\mathbf{\Delta}_g \circ \mathbf{S}_\mu$. Iterating this argument allows us to apply (6) recursively. This implies that the strongly coupled CML $\mathbf{\Delta}_g \circ \mathbf{S}_\mu$ reaches a banded state of the order of n from almost all initial conditions whenever $\mu \leq \bar{\mu}_n$. In particular, collective cycles of an arbitrarily large period must be reached for μ sufficiently close to μ_∞ and $g \geq g_e^*$. The actual collective dynamics exhibited though, depends on the behavior of the generalized CML $\mathbf{\Delta}_g^m \circ \mathbf{S}_\mu$.

We have performed numerical simulations of $\mathbf{\Delta}_g^m \circ \mathbf{S}_\mu$ for increasing m, for $d = 2$ and 3 lattices of coupled tent and logistic maps. In all cases, the asymptotic behavior observed for $m > 1$ is qualitatively the same as for $m = 1$ (Figs. 2a and 3a). Quantitative agreement is also very good, as one finds a fast convergence with m to a well-defined limit (insets of Figs. 2a and 3a). An immediate

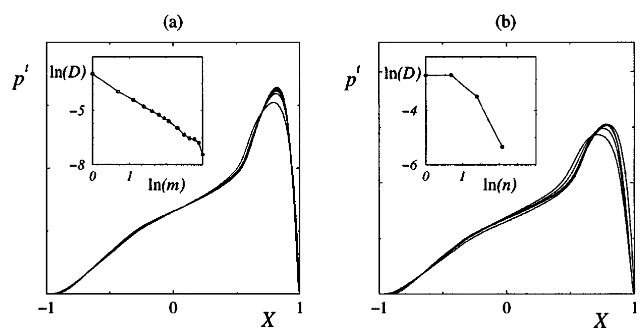

FIG. 3. (a) Same as Fig. 2 for coupled logistic maps. The convergence is to a nonuniversal distribution, because the RG for S_μ is not exact. (b) Central bands $I_{\bar{\mu}_n}^n$ of p^t at $\mu = \bar{\mu}_n$ transformed by the RG for $n = 0, 1, \ldots, 5$ (period-2^n NTCB). Convergence to a universal distribution representative of the continuous limit. Inset of (b): log-log plot of $D^2 = \int dX\,(p_{2^n} - p_{32})^2$ vs 2^n.

consequence of this observation is the qualitative band-by-band self-similarity of the bifurcation diagram of these CMLs, parallel to that of their local map S_μ. Quantitative agreement improves rapidly with decreasing μ, and is already excellent at relatively small orders.

In fact, statistical properties of CMLs are not expected to depend strongly on the coupling operator, provided it remains local [6]. It is a classic calculation to show that $\mathbf{\Delta}_g^m$ converges, for large m, to $\mathbf{\Delta}_\lambda^\infty = \exp(\frac{\lambda^2}{2}\nabla^2)$, a coupling operator with Gaussian weights where $\lambda = 2\sqrt{gm}\|\vec{e}\|$ is the typical coupling range with $\|\vec{e}\|$ the lattice mesh size (usually set to 1). For an infinite lattice, $\|\vec{e}\|$ plays no role while λ determines the size of the smallest structures. To be meaningful, the large m limit must therefore be taken at fixed $\lambda = \lambda_0$ and, consequently, $\|\vec{e}\| \to 0$. This constitutes the correct continuous limit of CMLs, i.e., a "field map" where the field $\mathbf{X}_{\vec{r}}$ with $\vec{r} \in \mathbb{R}^d$ evolves under the operator $\mathbf{\Delta}_{\lambda_0}^\infty \circ \mathbf{S}_\mu$. The asymptotic state is then unique, since λ is the unit length, and one can observe only the strong-coupling regime. In other words, $\lim_{m\to\infty} g_{\rm e}^*(m) = 0$. In the continuous limit, RG relation (6) reads

$$(\mathbf{\Delta}_{\lambda_0}^\infty \circ \mathbf{S}_\mu)^2 = \mathbf{h}_\mu^{-1} \circ \mathbf{\Delta}_{\lambda_0\sqrt{2}}^\infty \circ \mathbf{S}_{q(\mu)} \circ \mathbf{h}_\mu\,, \qquad (8)$$

which implies that length scales are renormalized by a factor of $\beta = \sqrt{2}$, independently of the local map [14]. Space-independent quantities such as p^t, on the other hand, scale with the Feigenbaum constants of the local map.

RG relation (8) is approximate only insofar as the RG of S_μ is approximate. It does not predict the behavior of the continuous CML on the interval $[\bar{\mu}_1, \bar{\mu}_0]$, but implies that, whatever this behavior may be, it is reproduced on intervals $[\bar{\mu}_{n+1}, \bar{\mu}_n]$ with an added 2^n periodicity and a rescaling of lengths by β^n. This proves that the cascades of phase transitions are infinite and that every critical point $\mu_n^{\rm c}$ observed in $[\bar{\mu}_{n+1}, \bar{\mu}_n]$ has counterparts in the higher-order μ intervals which converge with the exponent δ. Moreover, they must all share the same critical properties, since the characteristic divergence of the correlation length scales in the same way in $|\mu - \mu_n^{\rm c}|$ for all n. Again, all the properties of the continuous limit are, of course, only exact for tent maps; for more general maps, they are quantitatively valid in the $n \to \infty$ limit, in parallel to the status of the RG for S_μ itself.

RG relation (6) implies that all CMLs $\mathbf{\Delta}_g^m \circ \mathbf{S}_\mu$ converge to their continuous limit as $\mu \to \mu_\infty$ and/or $m \to \infty$. The results derived above are expected to apply in this limit, which, in practice, is reached very fast (Figs. 2 and 3). However, one can argue further that the critical properties of the phase transition points, where the correlation length diverges, are not expected to depend on the details of the coupling, and thus should all be the same for *any* μ and m. This is in agreement with the direct measurement of the critical exponents for the first three period-doubling phase transitions of $d = 2$ lattices of coupled logistic maps [9].

Some of the conclusions reached in the present work are identical to those reached in [6], but they are neither restricted to the weak coupling limit, nor to weakly inhomogeneous solutions. They explain the numerical observations of [7], where banded states were studied at rather large coupling ($g \simeq 0.1$) for $d = 2$ lattices of logistic maps. We believe our work is also relevant to the globally coupled case and provides the framework for a rigorous RG approach of systems of coupled maps. Finally, we would like to suggest that the continuous-space limit defined above constitutes an interesting system by itself, perhaps more tractable than CMLs for studying NTCB and spatiotemporal chaos. Since then $g_{\rm e} = 0$, this limit is not continuously related to the weak coupling regimes. This may explain the difficulties encountered when trying to extend weak coupling results to account for truly collective, strong-coupling behavior [15].

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[10] For $g = 0$ (no coupling), there are $\mathcal{N} = 2^N/2$ such frozen patterns. For small $g > 0$, $\mathcal{N}$ is smaller, but $\mathcal{N} \sim \exp(N)$, with N being the number of sites in the lattice. For $g > g_{\rm e}(\mu)$, $\mathcal{N}$ is small and *independent* of N [11].

[11] A. Lemaître and H. Chaté (to be published).

[12] The actual period could be larger than 2^n and/or combined with a quasiperiodic cycle.

[13] In fact, this usually happens for $g \le g_{\rm e}^*$.

[14] This explains, incidentally, why the numerical determination of the collective behavior is difficult in this region.

[15] See, e.g., J. Bricmont and A. Kupiainen, Commun. Math. Phys. **178**, 703 (1996); Physica (Amsterdam) **103D**, 18 (1997).