Quantum Dynamical Manifestation of Chaotic Behavior in the Process of Entanglement

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Manifestation of chaotic behavior is found in an intrinsically quantum property. The entanglement process, quantitatively expressed in terms of the reduced density linear entropy, is studied for the N-atom Jaynes-Cummings model. For a given energy, initial conditions are prepared as minimum uncertainty wave packets centered at regular and chaotic regions of the classical phase space. We find for short times a faster increase in decoherence for the chaotic initial conditions as compared to regular ones, which have oscillatory increase. [S0031-9007(98)06398-4]

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Classical Hamiltonian chaos is mostly studied in systems possessing 2 degrees of freedom. The investigation of the quantum counterparts of such systems is heavily based on their spectral properties and eigenfunctions [1], with a considerable amount of work concerning also quantum dynamical properties, e.g., dynamical localization [2]. Much less effort, however, has been dedicated to study the connection between quantum dynamical entanglement of parts of a larger system and chaos [3]. It has recently been proposed that the rate of entropy production can be used as an intrinsically quantum test of the chaotic versus regular nature of the evolution [4]. In the present Letter, we present a demonstration of this general point in the specific case of the N-atom Jaynes-Cummings model.

Let us assume that a quantum system is composed of two interacting subsystems. In order to study possible effects of the underlying chaotic dynamics in the process of entanglement of the subsystems, we must have some measure of the coherence loss of one of the subsystems. A measure of entanglement between two subsystems can be given by the linear entropy or the idempotency defect (or purity) of the subsystem of interest [5]. This can be calculated after evaluating the reduced density operator for the corresponding subsystem of interest (say subsystem 2)

$$\rho_2(t) = \mathrm{Tr}_1\{|\psi(t)\rangle\langle\psi(t)|\}. \tag{1}$$

From this operator we obtain the linear entropy as follows:

$$\delta(t) = 1 - \mathrm{Tr}_2\{\rho_2^2(t)\}. \tag{2}$$

In these equations $|\psi(t)\rangle$ is the full quantum state of the system and Tr_i means the trace over the variables of subsystem i (1 or 2). This quantity is clearly zero for the case where the subsystems are in pure nonentangled states. As we are especially interested in the entanglement process and not in the effect of statistical mixtures, we consider only initially pure and nonentangled states. Having chosen an initially separable state, as the time evolves we expect $\delta(t)$ to become nonzero as a result of the interaction. How fast it grows tell us how fast subsystem 2 suffers decoherence due to the entanglement with subsystem 1. Coherence loss has been important both theoretically and experimentally in various fields, ranging from the detection of gravitational waves to models in quantum optics [6]. The linear entropy, given by Eq. (2), has been used to study characteristic decoherence time scales in systems such as a harmonic oscillator coupled to a heat bath—much in use in condensed matter physics—as well as the Jaynes-Cummings model with importance in quantum optics and ion trap problems [7].

Recently much attention has been given to the spin-1/2 Jaynes-Cummings Model (JCM) in the context of cavity QED experiments [8]. Decoherence properties in an integrable situation have been measured [9]. Here, we explore the connection between chaos and entanglement, as explained above, by investigating in the N-atom JCM, how decoherence is affected by the presence of nonintegrable interactions. This model, for which the nonintegrable situation will probably become experimentally accessible in the near future [10,11], can be viewed as a subsystem of N two-level atoms coupled to a single-mode radiation field. In this example we show that the behavior of the linear entropy of the atomic subsystem is strongly correlated with the nature of the underlying classical motion in the following sense: Two wave packets with the same average energy, centered at regular and chaotic points in the classical phase space, exhibit very different short time behaviors for the entanglement process. The reason to choose the N-atom JCM, besides experimental application, is the fact that it has been extensively explored since its conception by Tavis and Cummings [12]. Chaotic behavior has been found semiclassically in the non-rotating-wave approximation (NRWA) [13]. Statistical properties of the quantum spectrum have been analyzed by several authors [14] and also scars of periodic orbits in Husimi distributions [15]. The connection between quantum and classical properties can be made only if the number of atoms is large enough ($N \geq 3$). Therefore, the study performed here will have $N = 9$, to make sure that we are treating a semiclassical regime.

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The correspondence of the quantum dynamical evolution and the classical phase space is made through the choice of the initial state. Given a point in the classical phase space, we construct the corresponding quantum initial conditions

$$|\psi(0)\rangle = |w\rangle \otimes |v\rangle, \tag{3}$$

where $|w\rangle$ stands for an SU(2) coherent state and $|v\rangle$ for a bosonic one [16]

$$|w\rangle = (1 + w\bar{w})^{-J} e^{wJ_+}|J, -J\rangle, \tag{4}$$

$$|v\rangle = e^{-v\bar{v}/2} e^{va^+}|0\rangle, \tag{5}$$

with [17]

$$w = \frac{p_1 + iq_1}{\sqrt{4J - (p_1^2 + q_1^2)}}, \tag{6}$$

$$v = \frac{1}{\sqrt{2}}(p_2 + iq_2), \tag{7}$$

$|J, -J\rangle$ being the state with spin J and $J_z = -J$, and $|0\rangle$ being the harmonic oscillator ground state. The full initial state $|\psi(0)\rangle$ is evolved by the quantum Hamiltonian

$$H = \omega_0 a^+ a + \varepsilon J_z + \frac{G}{\sqrt{2J}}(aJ_+ + a^+ J_-) + \frac{G'}{\sqrt{2J}}(a^+ J_+ + aJ_-), \tag{8}$$

where the first term corresponds to the energy of the free single-mode quantized field with frequency ω_0, described by the creation (annihilation) operators a^+ (a); the second term corresponds to the energy of the $N = 2J$ atoms with energy separation $\hbar\varepsilon$ ($\hbar = 1$ hereafter). The last two terms correspond to the interaction energy between the atomic system with the single-mode field. The second interaction term is the one responsible for the nonintegrability of the model. We then choose the atomic system as our subsystem of interest and analyze its decoherence process calculating the linear entropy given by Eq. (2).

The classical Hamiltonian corresponding to Eq. (8) can be written as a nonlinearly coupled 2 degrees of freedom function [18]

$$\mathcal{H}(q_1, p_1, q_2, p_2) = \frac{\omega_0}{2}(p_2^2 + q_2^2) + \frac{\varepsilon}{2}(p_1^2 + q_1^2) - \varepsilon J + \frac{\sqrt{4J - (p_1^2 + q_1^2)}}{\sqrt{4J}} \times (G_+ p_1 p_2 + G_- q_1 q_2), \tag{9}$$

where $G_\pm = G \pm G'$.

In Fig. 1 we show a Poincaré section of the classical counterpart for the spin degree of freedom, defined by the section $q_2 = 0$ in the four-dimensional phase space so that every time a trajectory pierces this section with $p_2 > 0$ the corresponding point (q_1, p_1) is plotted. In this figure we marked the initial conditions (i.c.) used as centers of the quantum coherent states to be evolved: i.c. belonging to stable islands are marked as filled circles, whereas the chaotic ones are marked by filled triangles. Figure 2 exhibits the corresponding curves of the exact calculation of $\delta(t)$. Note that the regular situation is dominated by a single frequency at short times [see Fig. 2(a)]. The chaotic situation reveals a comparatively much faster increase in decoherence. We have observed moreover that these two aspects interpolate smoothly (Fig. 3) in going from i.c. near periodic orbits to i.c. well inside the chaotic sea, thus having no sharp border between chaos and regularity. The existence of a plateau is interesting and persists for much longer times (we have checked the cases shown in Fig. 3 up to $t \approx 1000$). We believe this to be essentially related to the finiteness of the Hilbert space of the spin degree of freedom. In our calculation we have $2J + 1 = 10$ states. The initial state corresponds to a coherent state given in Eq. (4), so that there is an occupation probability for all states, concentrated around the center of the packet. As dynamics takes place, the relative population of the various available states tends to the one with an equal probability on the average. This conjecture is supported by the following numerical evidence: As J is increased, the value of $\delta(t)$ attained at the plateau increases accordingly.

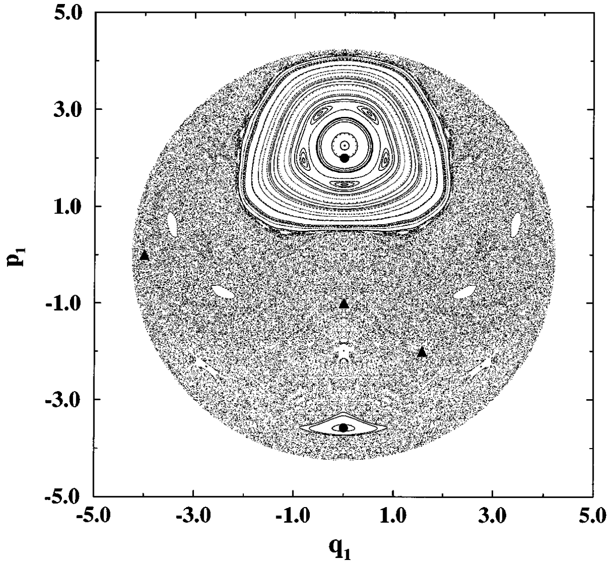

FIG 1. Poincaré section for the spin degree of freedom (section with $q_2 = 0.0$ and $p_2 > 0.0$) in the resonant case ($\varepsilon = \omega_0 = 1$), energy $E = 8.5$, $J = 9/2$ in a nonintegrable case ($G = 0.5$ and $G' = 0.2$). The marks represent the various choices for the center of the coherent states: filled circles for regular i.c. and filled triangles for chaotic i.c.

The main general aspects of decoherence and its relation to chaotic behavior from this particular example are the following: First, it is based on a quantity (linear entropy) that can be measured in any composite system; second, if there is an appropriate classical analog presenting soft chaos we can think that semiclassically the

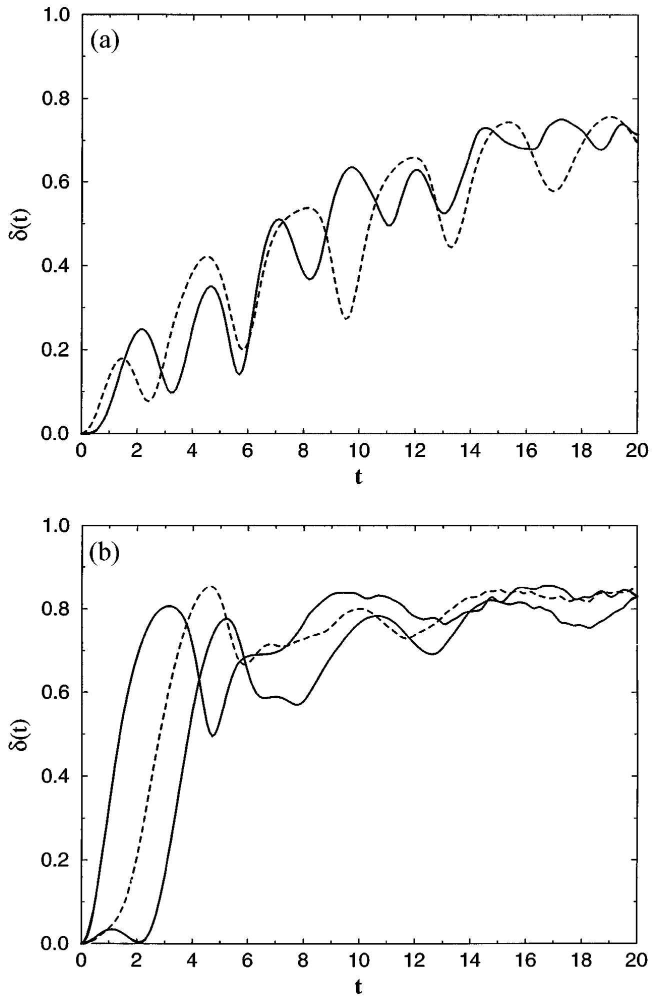

FIG. 2. Linear entropy $\delta(t)$ as a function of time in the resonant case ($\varepsilon = \omega_0 = 1$), mean energy $E = 8.5$, $J = 9/2$ in a nonintegrable case with $G = 0.5$ and $G' = 0.2$: (a) Regular i.c. corresponding to the filled circles shown in Fig. 1: ($q_1 = 0.0, p_1 = 2.0, q_2 = 0.0, p_2 = 3.615\,516$) for the continuous line and ($q_1 = 0.0, p_1 = -3.577, q_2 = 0.0, p_2 = 5.221\,772$) for the dashed line; (b) chaotic i.c. corresponding to the filled triangles shown in Fig. 1: ($q_1 = -4.0, p_1 = 0.0, q_2 = 0.0, p_2 = 3.162\,278$) for the fastest increase, ($q_1 = 0.0, p_1 = -1.0, q_2 = 0.0, p_2 = 5.724\,343$) for the next one (dashed line), and ($q_1 = 1.57, p_1 = -2.0, q_2 = 0.0, p_2 = 5.680\,464$) for the slowest of them.

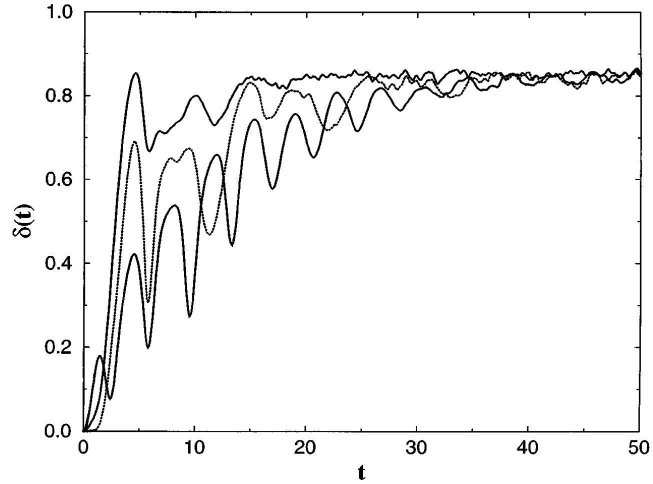

FIG. 3. Transition from regular to chaotic i.c. for the same parameters as in Fig. 2 with i.c. given by ($q_1 = 0.0, p_1 = -1.0, q_2 = 0.0, p_2 = 5.724\,343$) for the chaotic upper curve, ($q_1 = 0.0, p_1 = -2.0, q_2 = 0.0, p_2 = 6.084\,884$) for the "intermediary" dotted curve, and ($q_1 = 0.0, p_1 = -3.577, q_2 = 0.0, p_2 = 5.221\,772$) for the regular lower curve.

contributions for the wave packet from tori surrounding a regular i.c. involve always a limited well behaved region in phase space, and as a consequence a limited number of states are reached; whereas the nonperiodic chaotic orbits surrounding a chaotic i.c. do spread over all the chaotic sea in phase space, thus involving a greater number of states, and a dynamically faster entanglement between subsystems seems natural to occur. For larger dimensionality the diffusion in the phase space is facilitated because the KAM tori are no longer a restriction as in two-dimensional systems. A simple prediction from the present work is that the inclusion of nonintegrable interaction in an integrable Hamiltonian system with more than 1 degree of freedom (antiresonant term in the present case) should enhance decoherence. Moreover, even for a weak antiresonant term, an appropriate choice of the amplitude and phase of the atomic coherent state guided by the classical phase space chaos enables one to decrease or enhance decoherence.

In conclusion, we have found distinguished behavior for chaotic i.c. in an intrinsically quantum property. We think that an adequate way of investigating the problem of quantum chaos is by studying the dynamics of essentially quantum properties, of which the linear entropy (idempotency defect) is only one example. The connection between quantum and classical domains is done by means of the choice of coherent states, whose center is precisely on a classical phase space point. The time evolution will test the vicinities of this point. We believe that these results are applicable to more general systems in semiclassical regime corresponding to classical chaotic behavior: namely, if we choose an initial state corresponding to an integrable part of the phase space, the decoherence rate is slower than the one corresponding to a chaotic region. This prediction has been put forth by Zurek and Paz [4] in a very general context of a system coupled to an environment. They argue that in the weak dissipation limit the rate of entropy increase is dictated by the sensitivity to i.c. For a chaotic system the Wigner distribution associated with a phase space patch will be exponentially stretched in the unstable directions corresponding to positive Lyapunov exponents, while there will be a lower bound for the squeezing in the stable directions due to diffusion. By contrast for integrable systems this prediction changes qualitatively in the sense that the stretching is only polynomial. The resulting increase of the initial patch volume for both cases will mean an increase of the von Neumann entropy with a greater rate in the chaotic case. A bridge to the present work can be constructed by noting that the linear entropy $\delta(t)$ is a good approximation to the von Neumann entropy and, despite the fact that there is no coupling between our atomic subsystem and a real environment (in the sense of a bath, for example), we still have the mechanism of stretching in the unstable direction and squeezing in the stable one (with folding, moreover). In this context our results support the prediction by Zurek and Paz, which in turn reinforces the explanations for our findings.

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