Near-Critical Gravitational Collapse and the Initial Mass Function of Primordial Black Holes

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The recent discovery of critical phenomena arising in gravitational collapse near the threshold of black-hole formation is used to estimate the initial mass function of primordial black holes (PBHs). It is argued that the scaling relation between black-hole mass and initial perturbation found for a collapsing radiation fluid in an asymptotically flat space-time also applies to PBH formation in a Friedmann universe. Owing to the natural fine tuning of initial conditions by the exponential decline of the probability distribution for primordial density fluctuations, sub-horizon-mass PBHs are expected to form at all epochs. We derive a two-parameter mass function for PBHs. [S0031-9007(98)06434-5]

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In this Letter, we consider the initial mass function (IMF) of primordial black holes (PBHs) formed in the process of gravitationally collapsing primordial density fluctuations in the radiation dominated phase of the early Universe [1,2]. Implications of the PBH number and mass spectrum with regard to their contribution to the cosmic density and the γ-ray background (via Hawking evaporation) have been employed to constrain the spectral index of the primordial fluctuation spectrum [3,4]. Two aspects play a central role in these studies: first, for each horizon-sized space-time region there exists a critical threshold value, δ_c, for the density (or mass) contrast δ, separating its further evolution between formation of a black hole ($\delta > \delta_c$) and dispersion by pressure forces ($\delta < \delta_c$) (we shall use the term "horizon" to denote the particle horizon, $r_h \sim t$). Comparing the Jeans and horizon lengths at the time when the collapsing region breaks away from Hubble expansion, one finds that δ_c must be of order unity [5]. The second key assumption relates to the final mass of the black hole, M_{bh}. It is commonly assumed that M_{bh} is approximately equal to the mass of the collapsing region and thus to the horizon mass at the epoch of formation, M_h. Nevertheless, detailed predictions for the PBH IMF have not previously been made. Based on a scaling relation discovered in gravitational collapse of various near-critical space-times, generalized to collapsing density perturbations in an Einstein-de Sitter universe, we are able to derive a universal, two-parameter PBH IMF, applicable when PBH number densities are dominated from fluctuations collapsing during one particular epoch. Here the two parameters in the PBH IMF carry all the information of the statistics of the initial density spectra and the perturbation shapes. We show that when the perturbation overdensity is sufficiently close to the critical overdensity for PBH formation, δ_c, the final mass of the resulting PBH may be an arbitrarily small fraction of the horizon mass, providing a conceptual difference to our current understanding of the process of PBH formation.

It is possible that PBH formation is the only natural example for critical phenomena in gravitational collapse, a field of considerable interest in classical general relativity that was previously believed to have no astrophysical application. Triggered by the intriguing results of Choptuik [6] who demonstrated scaling and self-similarity in the gravitational collapse of a massless scalar field near the threshold of black-hole formation, critical phenomena were studied for a number of different setups, including spherically symmetric radiation fluids [7], Yang-Mills fields [8], and axially symmetric collapsing gravity waves [9]. In all cases, families of initial data quantified by a single generic parameter δ were found to give rise to a scaling relation of the form

$$M_{bh}(\delta) = K(\delta - \delta_c)^\gamma \qquad (1)$$

near the critical point for black-hole formation, δ_c. The specific choice of δ is arbitrary since differentiable transformations of δ leave (1) invariant, changing only the constant K to leading order [10]. Another noteworthy feature of near-critical solutions is the appearance of discrete (scalar field collapse) or continuous (perfect fluid collapse) self-similarities.

Equation (1) is, in general, irrelevant for the formation of astrophysical black holes. Degeneracy pressure of neutrons or electrons introduces intrinsic limiting mass scales of hydrostatic stability, such as the Chandrasekhar mass, violating the scale-free behavior indicated by Eq. (1). Moreover, Eq. (1) is valid only in the immediate neighborhood of δ_c, requiring a high degree of fine tuning of the initial conditions which is unnatural under most circumstances. In PBH formation, on the other hand, it is expected that most regions collapsing to a black hole will have overdensities close to the critical overdensity for PBH formation, δ_c, owing to a steeply declining probability distribution for primordial density fluctuations. Typical cosmic initial conditions thus provide the fine tuning of initial conditions, required for near-critical collapse. Further, with the exception of cosmological phase transitions that will not be considered here, the matter collapsing to PBHs is well described by a perfect fluid with radiation dominated equation of state, $p = \rho/3$, where p and ρ are pressure and energy density, respectively. Hence, the problem for PBH formation in radiation dominated cosmological epochs and the perfect fluid collapse studied numerically by Evans and Coleman (EC) [7] differ only with regard to the background space-time. While canonical initial conditions for PBH formation involve curvature perturbations in an expanding Friedmann-Robertson-Walker (FRW) space-time, EC used initial data embedded in an asymptotically stationary space-time for their collapse simulations.

In addition to their numerical simulations, EC found a self-similar solution to the equations of motion and gravitation in the limit $\delta \to \delta_c$. A self-similar ansatz reduces the spatial and temporal degrees of freedom to a single self-similar coordinate and thereby transforms the system of partial differential equations into ordinary ones. Demanding regularity at the center and along the ingoing acoustic characteristic, corresponding to the absence of a shock, the system of ODEs can be solved and the solution coincides well with their numerical results. As suggested by EC, the critical exponent of (1) was subsequently derived by analyzing linear perturbations of the self-similar solution: Koike, Hara, and Adachi [11] obtained $\gamma = 0.355\,801\,9$ for a collapsing radiation fluid. Note that neither the self-similar solution nor the perturbation analysis rely on asymptotic flatness of the space-time; on the contrary, EC's self-similar solution is not asymptotically flat. As EC's solution converges neither to a flat stationary space-time nor to an exact FRW solution, it invariably breaks down at large radii for both asymptotic behaviors.

The main reason to expect the emergence of self-similarity in near-critical gravitational collapse occurring in asymptotically FRW space-times is the separation of characteristic scales: Just as in the asymptotically stationary case studied by EC, the solution forms an intermediate asymptotic between two widely separated length scales [12]. The scale r_0 of the fluid perturbation δ at the onset of collapse is given by $\delta^{-1/2} r_h$ [5] if the initial perturbation amplitude is evaluated at horizon crossing. r_0 can be identified with the transition from Hubble expansion of the asymptotic FRW space-time to the collapse-dominated region $r < r_0$. On small scales, deviation from exact criticality leads to violation of self-similarity if r approaches $r_1 \sim K|\delta - \delta_c|$ [13]. The ratio r_0/r_1 can be made arbitrarily large by chosing initial data close to the critical point. In the limit $\delta \to \delta_c$, we therefore assume that gravitational collapse of a radiation fluid is well described by the self-similar solution of EC [7] and the critical exponent $\gamma \approx 0.356$ [11], independent of the asymptotic behavior of the background space-time. We note that preliminary results of numerical simulations of the PBH formation process in the early Universe confirm (1) the scaling relation and (2) the applicabilty of scaling for commonly assumed parameters of the statistics of pre-existing cosmic density fluctuations (see below). The results of this numerical investigation will be presented elsewhere [14].

Based on the arguments above we will henceforth employ Eq. (1) for the masses of PBHs formed by collapsing primordial density perturbations slightly exceeding δ_c, with an exponent $\gamma \approx 0.356$ independent of initial perturbation shape. We assume a Gaussian probability distribution for density fluctuations entering the horizon,

$$P(\delta) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{\delta^2}{2\sigma^2}\right), \tag{2}$$

where σ is the, possibly scale-dependent, root-mean-square fluctuation amplitude. Equation (2) allows us to compute the fraction of horizon-sized regions collapsing to PBHs at a given epoch [5]

$$\beta = \int_{\delta_c}^{1} P(\delta)\, d\delta \approx \sigma \exp\left(-\frac{\delta_c^2}{2\sigma^2}\right). \tag{3}$$

The upper integration limit reflects that if $\delta > 1$, the collapsing space-time region corresponds to a separate closed universe instead of a black hole [2], and the approximation on the right-hand side is valid to within a factor of a few for $\sigma/\delta_c \le 0.2$. It is noted that non-Gaussian effects may be important for $\delta_c \gg \sigma$ [15], but a Gaussian distribution suffices for the demonstration purpose of this work.

In what follows we assume that cosmological PBH formation is dominated by perturbations of one particular length scale, defining a characteristic epoch of PBH formation by the time the perturbations cross into the horizon. Such an analysis should be adequate when either the initial perturbation spectrum exhibits a peak on a given scale [16], or PBH formation is most probable during a specific epoch by virtue of the equation of state [17]. It is also approximately valid for blue initial perturbation spectra where PBH formation is most efficient on the smallest scale under consideration [18].

With these assumptions, and approximating incorrectly for the moment that the mass of the resulting PBH is M_h, we may compute the value of the PBH mass density divided by the cosmic background density,

$$\hat{\Omega}_{\mathrm{pbh,old}} \equiv \left\langle \frac{\rho_{\mathrm{bh}}}{\rho_0} \right\rangle = M_h^{-1} \int_{\delta_c}^{1} M_{bh} P(\delta)\, d\delta \approx \beta \quad \text{for } M_{bh} \approx M_h\,, \tag{4}$$

where the hat indicates that Ω_{pbh} is evaluated at the time of PBH formation.

As a straightforward modification of $\Omega\Omega_{pbh}$, we use the continuous distribution of PBH masses (1) in (4) and re-evaluate the integral. Doing so, we implicitly assume that (1) is valid for δ as large as unity; this need not necessarily be the case. However, the largest contribution to the integral comes from $\delta \approx \delta_c$ owing to the exponential form of $P(\delta)$, and thus our assumption is justified. The integrand rises steeply to a maximum at

$$\delta_m = \frac{1}{2}\left(\delta_c + \sqrt{4\gamma\sigma^2 + \delta_c^2}\right) = \delta_c + \frac{\gamma\sigma^2}{\delta_c} + O(\sigma^4) \quad (5)$$

close to the lower integration boundary, and the black-hole mass at this point is

$$M_{bh}(\delta_m) = K\left(\frac{\gamma\sigma^2}{\delta_c}\right)^{\gamma} \approx k\sigma^{2\gamma}M_{\mathrm{h}}, \quad (6)$$

with the dimensionless k defined by $K = kM_h$. The modified expression for $\hat{\Omega}_{pbh}$ is thus

$$\begin{aligned}\hat{\Omega}_{pbh,\mathrm{new}} &= M_h^{-1}\int_{\delta_c}^{1} M_{bh}(\delta)\,P(\delta)\,d\delta \\ &\approx M_h^{-1}\int_{\delta_m}^{1} M_{bh}(\delta)\,P(\delta)\,d\delta \\ &\approx k\sigma^{1+2\gamma}\exp\left(-\frac{\delta_c^2}{2\sigma^2}\right) \approx k\sigma^{2\gamma}\beta, \quad (7)\end{aligned}$$

where the integral was asymptotically expanded to first order. Equation (7) shows that the average black-hole mass produced at each epoch is approximately given by (6), since

$$\langle M_{bh}\rangle = \beta^{-1}\int_{\delta_c}^{1} M_{bh}(\delta)\,P(\delta)\,d\delta \approx k\sigma^{2\gamma}M_h. \quad (8)$$

We can now determine the PBH initial mass function (IMF) when PBH number densities are dominated from formation during one particular epoch. The global PBH mass spectrum generally involves an integration over all epochs, a formidable problem, which will not be attempted here. We define the PBH IMF as the fraction $d\phi$ of PBH number per logarithmic mass interval, normalized such that

$$\int_{-\infty}^{\ln M_{bh}(\delta=1)} \frac{d\phi}{d(\ln M'_{bh})}\,d(\ln M'_{bh}) = 1. \quad (9)$$

This mass function is given by

$$\begin{aligned}\frac{d\phi}{d(\ln M_{bh})} &= \beta^{-1}P[\delta(M_{bh})]\,\frac{d\delta}{d(\ln M_{bh})} \\ &= \frac{1}{\sqrt{2\pi}\,\beta\sigma\gamma}m_{bh}^{1/\gamma}\exp\left(-\frac{(\delta_{\mathrm{c}} + m_{bh}^{1/\gamma})^2}{2\sigma^2}\right), \quad (10)\end{aligned}$$

where m_{bh} is black hole mass in units of kM_h, and where we have used Eq. (2) for $P(\delta)$. The PBH IMF of Eq. (10) has wider applicability than naively thought. Imagine PBH formation in the case of non-Gaussian statistics, in particular, when $P(\delta)$ is different from Eq. (2). In this case one may search for a control parameter $\delta'(\delta)$ which renders $P(\delta)d\delta/d\delta'$ Gaussian. Applying this transformation between control parameters to Eq. (1), one will obtain a form-invariant equation (1) with modified constants K' and δ'_c, provided the limit of near-critical gravitational collapse still applies. Equation (10) then defines a universal two-parameter family of PBH IMFs, applicable for many initial conditions, with the parameters K and δ_c carrying all the information about the statistics of initial conditions and the shapes of perturbations.

The mass function of Eq. (10) exhibits a maximum at

$$M_{bh}^{\max} = k\left(\frac{\sigma^2}{\delta_c}\right)^{\gamma}M_h, \quad (11)$$

which approximately equals the average black-hole mass equation (8). PBHs in cosmological interesting numbers are formed during the evolution of the early Universe for values of $\sigma/\delta_c \approx 0.1 - 0.2$, provided Gaussian statistics holds [4]. Such values of σ/δ_c yield typical volume collapse fractions in the range $\beta \approx 10^{-6}$–10^{-23} and, depending on the epoch of formation, imply PBH number densities significantly contributing to the present mass density, or the γ background. Inserting $k \approx 3.3$ found by EC [19], $\sigma/\delta_c \approx 0.15$, and $\delta_c \approx 1/3$ [5], we find

$$M_{bh}^{\max} \approx 0.6M_h. \quad (12)$$

It is not surprising that the maximum of the IMF at a fixed epoch coincides with the horizon mass to within an order of magnitude, since the latter determines the mass scale for collapse. However, depending on the value of σ, a fraction of all PBHs formed at each epoch will have masses significantly smaller than M_h, implying a fundamental conceptual difference between this work and previous calculations. It was previously assumed that there exists a one-to-one correspondence between M_{bh} and redshift z. Under this assumption, it was straightforward to relate M_{bh} to a single energy scale, i.e., microscopically small black holes only formed at very early times. Using Eq. (1) instead, this simplification is no longer valid; the formation of black holes with a continuous IMF allows the formation of microscopic PBHs at all epochs. The formulation of observational constraints based on these results, such as constraints on the spectral index of initial density spectra, requires a detailed analysis of the PBH IMF integrated over all epochs which is beyond the scope of this Letter.

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[13] We use $G = c = 1$ throughout the paper such that K has dimensions of mass and length. For example, the horizon mass is $M_h = (4\pi/3)\rho r_h^3 = r_h/2$.
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[18] Clearly, this analysis is incorrect for scale-invariant perturbation spectra. Nevertheless, given the COBE normalization of $\sigma \approx 10^{-5}$ on large scales and the assumption of Gaussianity, scale invariance would imply that PBH formation in the early Universe is of no cosmological importance.
[19] Doing so, we make a specific choice for the control parameter δ. Evans and Coleman defined $\delta = 2M/r_0$. This corresponds in our case to the mass that is contained within the horizon sphere in addition to the unperturbed FRW horizon mass. We note here that for canonical Gaussian density perturbations δ in Eq. (2) is the average overdensity in uniform Hubble constant gauge determined in the limit of linear evolution, which may yield a different value of k. Further, k may be perturbation shape dependent such that an exact evaluation of this quantity has to await the results of numerical simulations.