Vortex Formation in Two Dimensions: When Symmetry Breaks, How Big Are the Pieces?

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We investigate the dynamics of second-order phase transitions in two dimensions, breaking a gauged U(1) symmetry. Using numerical simulations, we show that the density of the topological defects formed scales with the quench time scale τ_Q as $n \sim \tau_Q^{-1/2}$ when the dynamics is overdamped at the instant when the freeze-out of thermal fluctuations takes place, and $n \sim \tau_Q^{-2/3}$ in the underdamped case. This is predicted by the scenario proposed by one of us (W. H. Z.). [S0031-9007(98)06423-0]

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Phase transitions occur at all energy scales, from Bose-Einstein condensation near absolute zero to the sub-Planck temperatures relevant for symmetry breaking in a cosmological setting. While the dynamics of some types of first-order transitions (i.e., the process of nucleation) has been extensively studied, analogous understanding of second-order transitions is emerging only now. In this case there is no supercooling and the final state of the system is asymptotically approached through phase ordering. Until recently, research has focused largely on the asymptotic scalings of this process [1] rather than on the dynamics of the transition itself.

The change in focus is relatively recent. Kibble [2] pointed out that topological defects may have significant cosmological consequences—i.e., they may act as seeds for structure or as constraints on models. While their initial statistics may be wiped out at later times, some features (such as the *ab initio* existence of an "infinite" string) are essential for the scenario. Also, the initial defect density may be directly relevant for generation of baryons [3].

The configuration of the order parameter established in the course of the transition is therefore important. Furthermore, it is accessible in cases where topological defects are formed, as they bear witness to the dynamics of the order parameter in the vicinity of the critical point. Experiments based on this idea allow one to probe the critical dynamics of symmetry breaking, and have been carried out in liquid crystals [4] and in superfluids [5–8]. They have led to new insights into the dynamics of the transition between phases A and B of ^{3}He [9]. A description of the dependence of defect density on the rate of the transition and the damping of the order parameter in second-order transitions has been proposed by one of us [10], and in this Letter we discuss further and verify this model with numerical simulations.

We study the consequences of second-order phase transformations of the order parameter ψ, a complex scalar, with Landau-Ginzburg dynamics in two spatial dimensions and an associated gauge field A_a. This is the Abelian Higgs model coupled to a heat bath and with a dissipative term. When cast into first-order form, the equations of motion are the following [$\psi_X = (\psi_1, \psi_2)$, i runs over $\{x, y\}$, and $D_a\psi = \partial_a + ieA_a\psi$]:

$$\pi_X = \partial_t \psi_X, \qquad \Pi_i = \partial_t A_i, \tag{1}$$

$$\partial_t \pi_X = \nabla^2 \psi_X - e^2 A^2 \psi_X - 2e\epsilon_{XY} A^i \partial_i \psi_Y - \frac{\partial V}{\partial \psi_X} - \eta \pi_X + \vartheta, \tag{2}$$

$$\partial_t \Pi_i = \nabla^2 A_i - e\epsilon_{XY} \psi_X \partial_i \psi_Y - e^2 A_i |\psi|^2, \tag{3}$$

$$V(\psi) = -\frac{1}{2}\epsilon m_0^2 \psi^2 + \frac{1}{4}\psi^4. \tag{4}$$

The system is subjected to the white noise $\vartheta(x, y, t)$; $\langle \vartheta(x, y, t)\vartheta(x', y', t')\rangle = 2\eta\theta\delta(x - x')\delta(y - y')\delta(t - t')$. The heat bath temperature is θ and η sets the damping rate, in accordance with the fluctuation-dissipation theorem.

We induce the symmetry breaking by changing the sign of the dimensionless parameter ϵ in the effective potential over the quench time scale τ_Q, so that $\epsilon = t/\tau_Q$ ($|\epsilon| \leq 1$). The phase transition occurs when it becomes energetically and entropically favorable for the order parameter to assume (in equilibrium) a finite expectation value. In our case this happens in the region $0 < \epsilon \ll 1$. The shift of critical temperature from $\epsilon = 0$ is due to the coupling to the gauge field ($m_{\rm eff}^2 \sim m_0^2 + e^2\langle A^2\rangle$) and as a result of finite temperature θ, which we take to be 0.01.

The equations of motion are evolved on a torus of 512^2 grid points, using the staggered leapfrog method. We use the gauge $A_t = 0$, implicit in Eqs. (2) and (3), and perform 20 realizations of each parameter set. We resolved defects by several grid spacings at low temperatures.

We begin the simulations well above the phase transition, allowing the system to equilibrate under the influence of the noise and relaxation at constant ϵ. We monitor the order parameter and the gauge field throughout the subsequent evolution. The focus of attention, however, is the number of topological defects, which can be identified as zeros of ψ in the broken symmetry phase. Initially, in the

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symmetric phase, such zeros are plentiful and short lived (see Fig. 1). While they cannot be identified with defects, their density and arrangement gives an idea of the nature of the thermal fluctuations.

Below T_c their density decreases, although there still exist regions over which the field has a near-zero expectation value and hence many unstable zeros. Eventually, only a few isolated defects remain. We count only those that have no companions within $\xi_0 = m_0^{-1}$.

The local and global ($e = 0$) gauge cases are qualitatively indistinguishable until this late stage. However, local defects do not interact over more than a few correlation lengths and annihilate relatively slowly, even when the friction coefficient is small. By contrast, global defects interact with a logarithmic potential, and disappear much more rapidly. Estimates of initial densities become more difficult in this case. Below, we shall focus on the local case, leaving global-local comparisons for the future.

The theory of defect formation combines the realization, due to Kibble [2], that the domains of the order parameter ψ which break symmetry incoherently must contain of the order one "fragment" of a defect, with the estimate [10] of the relevant size based on the comparison of the relaxation time scale of ψ with the effective rate of change of the mass parameter ϵ. In the immediate vicinity of the critical temperature, the dynamics of ψ are subject to critical slowing down. Thus, the time scale τ over which the order parameter can react is given by $\tau_{\dot{\psi}z} = \eta\tau_0^2/|\epsilon|$ and $\tau_{\ddot{\psi}} = \tau_0/|\epsilon|^{1/2}$ ($\tau_0 = 1/m_0$) in the overdamped and underdamped cases, respectively, where correspondingly the first or second time derivative in Eq. (2) dominates. The overdamped scenario is presumably more relevant for condensed matter applications, while in the cosmological settings ψ may be underdamped.

The characteristic time scale of variations of ϵ is $\epsilon/\dot{\epsilon} = t$. The system is able to readjust to the new equilibrium as long as the relaxation time is smaller than t. Hence, outside the interval $[-\hat{t}, \hat{t}]$ defined by $\tau[\epsilon(\hat{t})] = \hat{t}$, the evolution of ψ is approximately adiabatic, and physical quantities follow their equilibrium expectation values. Thus the times $\hat{t}_{\dot{\psi}} = \pm\tau_0\sqrt{\eta\tau_Q}$ and $\hat{t}_{\ddot{\psi}} = \pm m_0^{-2/3}\tau_Q^{1/3}$, giving $\hat{\epsilon}_{\dot{\psi}} = \pm(\eta\tau_0^2/\tau_Q)^{1/2}$ and $\hat{\epsilon}_{\ddot{\psi}} = \pm(m_0^2/\tau_Q)^{2/3}$, mark the borders between the (approximately) adiabatic and impulse (or "drift") stages of evolution of ψ in the overdamped and underdamped cases, respectively.

In particular, the correlation length ξ of ψ above the transition will cease to follow the Landau-Ginzburg scaling ($\xi = \xi_0/|\epsilon|^{1/2}$) once the adiabatic-impulse boundary at $-\hat{t}$ is reached. Dynamics will be suspended (except for the drift and noise) in the interval $[-\hat{t}, \hat{t}]$ and will resume at $+\hat{t}$ below the transition.

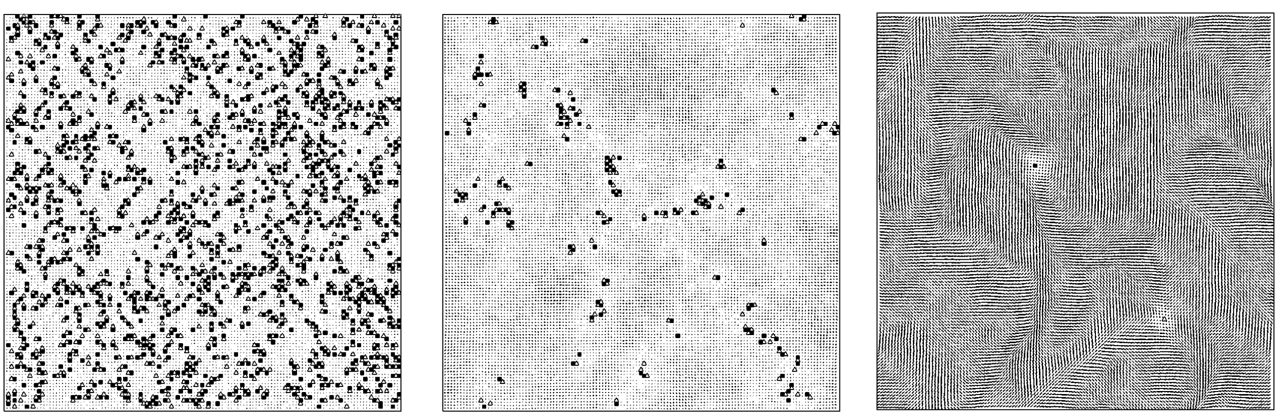
We expect, then, that the characteristic length scale over which ψ is ordered already in the course of the transition will be the correlation length at freeze-out, $\hat{\xi} = \xi_0/\sqrt{\hat{\epsilon}}$ [10]. This results in $\hat{\xi}_{\dot{\psi}} = \xi_0(\tau_Q/\eta\tau_0^2)^{1/4}$ and $\hat{\xi}_{\ddot{\psi}} = \xi_0(\tau_Q/\tau_0)^{1/3}$ in the two cases. The initial density of vortex lines in two dimensions should then scale as

$$n_{\dot{\psi}} = \frac{1}{(f_{\dot{\psi}}\hat{\xi}_{\dot{\psi}})^2} = \frac{1}{(f_{\dot{\psi}}\xi_0)^2}\sqrt{\frac{\eta\tau_0^2}{\tau_Q}}, \tag{5}$$

$$n_{\ddot{\psi}} = \frac{1}{(f_{\ddot{\psi}}\hat{\xi}_{\ddot{\psi}})^2} = \frac{1}{(f_{\ddot{\psi}}\xi_0)^2}\left(\frac{\tau_0}{\tau_Q}\right)^{2/3} \tag{6}$$

in the overdamped and underdamped regimes, respectively. Above, f is the proportionality factor which is expected to be of the order of a few [11], and which may be estimated analytically in some cases [12].

The relative importance of the $\dot{\psi}$ and $\ddot{\psi}$ terms in Eq. (2) also depends on $\hat{\epsilon}$. What matters for the formation of defects is—in view of the arguments above—which of the two terms dominates at $\hat{t}$. Thus, $\eta^2 > \hat{\epsilon}$ is the condition for overdamped evolution, which leads to the inequality $\eta^3\tau_Q > m_0^2$. Therefore, one can enter the overdamped regime by performing a sufficiently slow quench, as well as increasing the damping parameter η.



(a) (b) (c)

FIG. 1. (a)–(c) Representations of the field and defects (filled squares or open triangles) on 128^2 grid during and after a quench of time scale $\tau_Q = 32$, with $\eta = 1.0$ and in the presence of a gauge field ($e = 0.5$). (a) shows the field at the critical point ($\epsilon = 0$), (b) at $\epsilon = 0.6$, and (c) at $\epsilon = 1$, time $t = 2\tau_Q$ after the critical point.

We verify these scalings in Fig. 2, where the numbers of defects obtained in both the overdamped and underdamped cases are plotted as a function of the quench time τ_Q. In the overdamped case the annihilation of the well-separated topological defects is slow. Consequently, it is relatively easy to count them at some fixed time. This yields Fig. 2a with $n \sim \tau_Q^\gamma$, $\gamma = -0.44 \pm 0.1$ for $\tau_Q \geq 8$, in good agreement with the theoretical prediction of $\gamma = -1/2$ [10]. A similar conclusion can be reached for the underdamped case, Fig. 2b, except that annihilation is now quite rapid, and the initial number of defects is harder to define. This is especially true for the fastest quenches which produce most defects. However, when the three left-most points most affected by annihilation are ignored, straightforward counting of vortices yields $\gamma = -0.6 \pm 0.07$. This is in agreement with the theory—which in this case predicts $\gamma = -2/3$ [10,13]—and with the indications from kink formation in one dimension [14].

The reason the slope might become shallower for small τ_Q is easy to understand. A very fast quench becomes indistinguishable from an instantaneous one, which does not allow for the adiabatic regime we have noted above. Instantaneous quenches start the evolution in the broken symmetry phase, but with an initial field configuration which will contain many zeros per healing-length size volume—too many to regard them as well-defined defects. Unless the defects are well separated at their conception (which in effect requires $\hat{\epsilon} \ll 1$), annihilation will decide their initial density. We see this already in Fig. 2a, where $\hat{\epsilon} = 0.5$ for the leftmost point. The effect is even more pronounced in the underdamped case, which allows for more rapid annihilation (Fig. 2b).

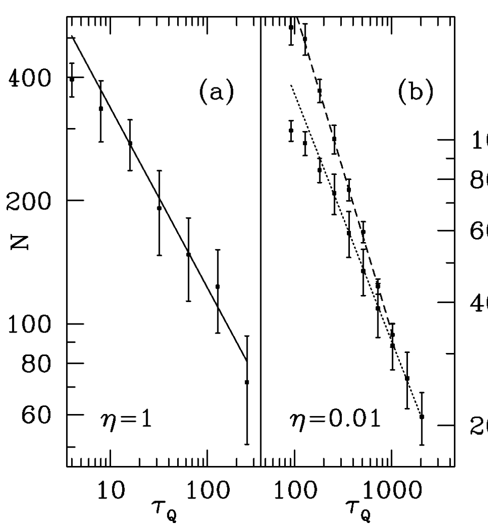


FIG. 2. The variation of defect count N with τ_Q in both (a) overdamped and (b) underdamped regimes. The fitted slopes are (a) -0.44 ± 0.10 ($\chi^2 = 0.44$, dropping the two leftmost points) and (b) -0.60 ± 0.07 ($\chi^2 = 0.034$, dropping three points). Predicted values were $-1/2$ and $-2/3$. The points fitted by the dashed line are inferred from the fits in Fig. 3. The slope is -0.79 ± 0.04 ($\chi^2 = 0.27$).

The annihilation rate is expected to be of the form $\dot{n} = -\chi(\eta) n^2$, as it is proportional to the frequency with which defects encounter one another. This leads to

$$n^{-1} = n_0^{-1} + \chi t \,. \tag{7}$$

Figure 3 shows (in the underdamped case, when the annihilation is appreciable) the fit between (7) and the data. We can then infer the value of the initial defect density n_0, which is also shown in Fig. 2b. This procedure (followed, e.g., in the superfluid work [5]) yields a somewhat different slope $\gamma = -0.79 \pm 0.04$, steeper than equal-time data, and somewhat steeper than the theoretical prediction. It also seems to successfully correct for the annihilation (although the leftmost point still seems to be affected). Using the data in Fig. 2, we estimate $f_{\dot{\psi}} \simeq 12$ and $f_{\ddot{\psi}} \simeq 8$ (equal-time data), $f_{\ddot{\psi}} \simeq 4$ [fitting to Eq. (7)].

One of the issues in defect formation is the significance of the thermal fluctuations, which continue to rearrange the order parameter configuration down to the Ginzburg regime (i.e., below the transition temperature). If, as was once thought, fluctuations and Ginzburg temperature were to determine initial defect density, then heating the system above the critical point would presumably erase the pre-existing configuration of defects and create a new one. We performed a numerical experiment to test the importance of fluctuations. The initial configuration with defects present is the one shown in Fig. 1c. We reheat this, varying ϵ from the broken symmetry value $\epsilon = 1$ to values around the critical point $\epsilon \simeq 0$, using the same τ_Q as in the original quench of Fig. 1.

The results (Fig. 4) demonstrate that even when the system is taken above the critical point, the initial configuration of defects is eventually recovered, as long as the reheating does not take it further than $\hat{\epsilon}$ into the symmetric phase. This confirms the theory put forward in [10],

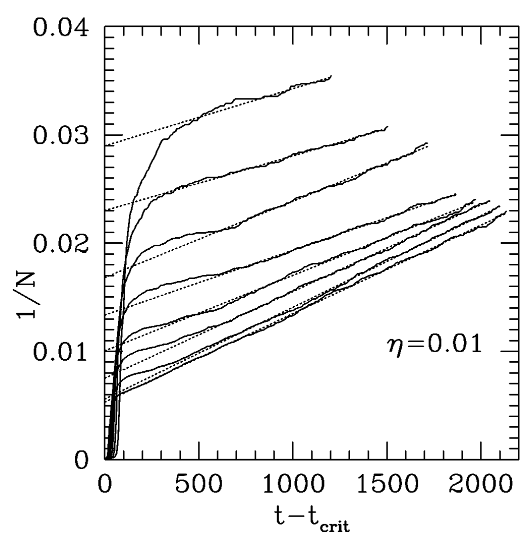


FIG. 3. Defect annihilation in the underdamped regime. The inverse of the defect count is plotted against time for various τ_Q. Also shown are least-squares fits using Eq. (7).

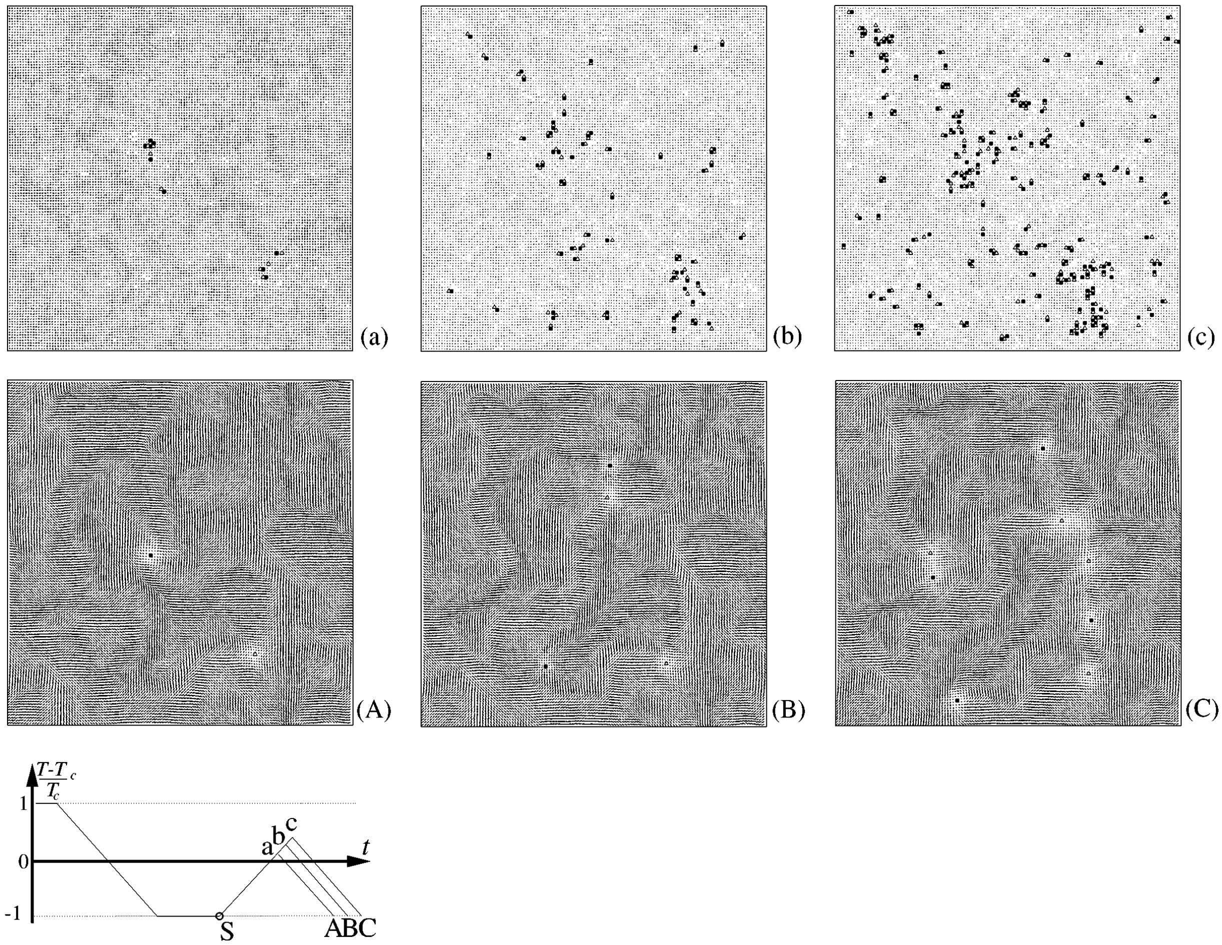


FIG. 4. Memory and reheating: The post-transition configuration shown in Fig. 1 (the point S in the sketch to the left) is reheated to various temperatures close to the critical point (a)–(c) and cooled again to $\epsilon = 1$ [(A)–(C), respectively]. The reheat values of ϵ are (a) -0.1, (b) -0.2, and (c) -0.25. Here $|\hat{\epsilon}| \simeq 0.17$. Notice that the memory of the configuration is largely preserved even when the critical temperature is exceeded during reheating [(A) and (B)]. Memory is erased only when the temperature crosses the $|\epsilon|$ freeze-out zone associated with the formation of the original defect configuration.

and leads one to conclude that thermal fluctuations cannot significantly rearrange configurations of the order parameter on scales larger than $\sim\hat{\xi}$, unless the "impulse strip" $|\epsilon| < |\hat{\epsilon}|$ is traversed (or unless the time spent in that regime is set by a time scale other than τ_Q).

We have used high-resolution numerical simulations to explore phase transitions in two dimensions and have found the scaling with quench time scale and damping agree with the predictions of the Kibble-Zurek scenario. The importance of the freeze-out time $\hat{t}$ as the defining moment for defect formation has been illustrated.

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