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Mixed-State Entanglement and Distillation: Is there a "Bound" Entanglement in Nature?

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It is shown that if a mixed state can be distilled to the singlet form it must violate partial transposition criterion [A. Peres, Phys. Rev. Lett. **76**, 1413 (1996)]. It implies that there are two *qualitatively* different types of entanglement: "free" entanglement which is distillable, and "bound" entanglement which cannot be brought to the singlet form useful for quantum communication purposes. A possible physical meaning of the result is discussed. [S0031-9007(98)06051-7]

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Since the famous Einstein, Podolsky, and Rosen [1] and Schrödinger [2] papers, quantum entanglement still remains one of the most striking implications of quantum formalism. In recent years, great effort has been made to understand the role of entanglement in nature and fundamental applications were found in the field of quantum information theory [3–7]. The most familiar example of pure entangled state is the singlet state [8] of two spin-$\frac{1}{2}$ particles,

$$\Psi_- = \frac{1}{\sqrt{2}} (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle), \tag{1}$$

which cannot be reduced to a direct product by any transformation of the bases pertaining to each of the particles.

In practice, due to decoherence effects, we usually deal with mixed states [9]. A mixed state of a quantum system consisting of two subsystems is supposed to represent entanglement if it is inseparable [10], i.e., cannot be written in the form

$$\varrho = \sum_i p_i \varrho_i^A \otimes \varrho_i^B, \qquad p_i \geq 0, \qquad \sum_i p_i = 1, \tag{2}$$

where ϱ_i^A and ϱ_i^B are states for the two subsystems. However, to use the entanglement for quantum information processing, we must have it in pure singlet form. The procedure of converting mixed state entanglement to the singlet form is called distillation [11]. It amounts to the extraction of pairs [12] of particles in singlet state from an ensemble described by some mixed state by means of local quantum operations and classical communication [11].

The process can be described as follows: The two observers, Alice and Bob, each have N quantum systems coming from entangled pairs prepared in a given state ρ. Each one can perform local operations with her/his N particles, and exchange classical information with the other. The question is whether they can in this way obtain a pair of entangled qubits in nearly singlet state (the rest of the quantum systems being discarded). They need not succeed every time, but at least they know when they have been successful. If they managed to do this, one says that they have *distilled* some amount of pure entanglement from the state ϱ. Subsequently, the distilled singlet pairs can be used, e.g., for reliable transmission of quantum information via teleportation [5] (experimental realizations of quantum teleportation have been recently reported, see Ref. [6]).

Recently, it has been shown [13] that *any* inseparable two-qubit state [14] represents the entanglement which, however small, can be distilled to a singlet form. The result was obtained by use of the necessary [15] and sufficient [16] condition of separability for two-qubit states, local filtering [17,18], and Bennett *et al.* distillation protocol [11].

In this context it seems very natural to make the following conjecture.

Conjecture.—Any inseparable state can be distilled to the singlet form.

Surprisingly enough, this conjecture is wrong. In the present Letter we will show that there are inseparable states that *cannot* be distilled. More specifically, we first show that any state which can be distilled must violate Peres separability criterion [15]. Then, the result follows from the fact [19] that there are inseparable states that satisfy the criterion. It shows that there are two qualitatively different types of entanglement. The first, "free" entanglement, can be distilled to the singlet form. The second type of entanglement is not distillable and is considered here in analogy with thermodynamics as a "bound" entanglement which cannot be used to perform a useful "informational work" such as reliable transmission of quantum data via teleportation.

Now, let us first briefly describe the Peres criterion. A state ϱ satisfies the criterion, if all eigenvalues of its partial transposition ϱ^{T_B} are non-negative (i.e., if ϱ^{T_B} is a positive operator). Here, the partial transposition ϱ^{T_B} associated with an arbitrary product orthonormal $e_i \otimes f_j$ basis is defined by the matrix elements in this basis:

$$\varrho^{T_B}_{m\mu,n\nu} \equiv \langle e_m \otimes f_\mu | \varrho^{T_B} | e_n \otimes f_\nu \rangle = \varrho_{m\nu,n\mu} . \tag{3}$$

Clearly, the matrix ϱ^{T_B} depends on the basis, but its eigenvalues do not. Thus, given a state, one can check whether it violates the criterion performing the partial transposition in an arbitrary product basis. In particular, it implies that ϱ violates the criterion if and only if any N-fold tensor product $\varrho^{\otimes N} = \underbrace{\varrho \otimes \ldots \otimes \varrho}_{N}$ does [15].

Peres showed that the criterion must be satisfied by any separable state [15]. It has also been shown [16] that for two-qubit (and qubit-trit) states the criterion is also *sufficient* condition for separability. This does *not* hold for higher dimensions. The explicit examples of inseparable mixtures satisfying criterion were constructed [19].

Now, we are in a position to present the main result of this Letter. Suppose Alice and Bob have a large number N of pairs each in a state ϱ acting on the Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$. Then, the joint state of N pairs is given by $\varrho^{\otimes N}$. Suppose now that the state ϱ is distillable. This means that Alice and Bob are able to obtain pure singlet two-qubit pairs for N tending to infinity. This, however, implies that, for some finite N, they are able to obtain an inseparable two-qubit state $\tilde{\varrho}_{2q}$. The most general operation producing a two-qubit pair that can perform over the initial amount of N pairs can be written in the following form [20]:

$$\tilde{\varrho}_{2q} = \frac{1}{M} \sum_i A_i \otimes B_i \varrho^{\otimes N} A_i^\dagger \otimes B_i^\dagger , \tag{4}$$

where $M = \mathrm{Tr} \sum_i A_i \otimes B_i \varrho^{\otimes N} A_i^\dagger \otimes B_i^\dagger$ is the normalization factor, and A_i and B_i map the large Hilbert spaces $\mathcal{H}_{A,B}^{\otimes N}$ into C^2. For convenience, we will use unnormalized states, as the property of separability as well as satisfying the Peres criterion do not depend on the positive factor. Then, for unnormalized states, we omit the condition $\sum_i p_i = 1$ in the definition of separability (2). Consequently, let

$$\varrho_{2q} = \sum_i A_i \otimes B_i \varrho^{\otimes N} A_i^\dagger \otimes B_i^\dagger \tag{5}$$

and

$$\varrho_i = A_i \otimes B_i \varrho^{\otimes N} A_i^\dagger \otimes B_i^\dagger . \tag{6}$$

Since ϱ_{2q} is inseparable then at least for some $i = i_0$ the state ϱ_{i_0} must be inseparable. Indeed, by summing separable states we cannot get an inseparable one.

Note that the operators A_{i_0} and B_{i_0} act in two-dimensional space C^2, hence they can be written in the form

$$\begin{aligned} A_{i_0} &= |0\rangle\langle\psi_A| + |1\rangle\langle\phi_A| , \\ B_{i_0} &= |0\rangle\langle\psi_B| + |1\rangle\langle\phi_B| , \end{aligned} \tag{7}$$

where $|1\rangle$ and $|0\rangle$ constitute the orthonormal basis in C^2 and $\psi_A, \phi_A \in \mathcal{H}_A^{\otimes N}$, $\psi_B, \phi_B \in \mathcal{H}_B^{\otimes N}$ are arbitrary (possibly unnormalized) vectors. Let us now consider two-dimensional projectors P_A and P_B which project onto the spaces spanned by ψ_A, ϕ_A and ψ_B, ϕ_B, respectively. Then, we have

$$\varrho_{i_0} = A_{i_0} \otimes B_{i_0} (P_A \otimes P_B \varrho^{\otimes N} P_A \otimes P_B) A_{i_0}^\dagger \otimes B_{i_0}^\dagger . \tag{8}$$

Now, since a product action cannot convert a separable state into an inseparable one, we obtain that also the state

$$\varrho' = P_A \otimes P_B \varrho^{\otimes N} P_A \otimes P_B \tag{9}$$

is inseparable. Let us write this state in basis $|f_i\rangle \otimes |g_k\rangle$, $i = 1, 2, \ldots, \dim\mathcal{H}_A^{\otimes N}$, $k = 1, 2, \ldots, \dim\mathcal{H}_B^{\otimes N}$ with four vectors $|f_1\rangle, |f_2\rangle$ ($|g_1\rangle, |g_2\rangle$) spanning the subspaces defined by projectors P_A, P_B. The only nonzero matrix elements are due to products of those vectors and they define a 4×4 matrix M_{2q} which can be thought of as a two-qubit state. The operation of partial transposition on ϱ' affects only those elements (as the remaining ones are equal to zero). If M_{2q} were positive after partial transposition, then, due to the sufficiency of the partial transposition test for the two-qubit case [16], M_{2q} would represent a separable two-qubit state. Hence, if embedded into the whole space $\mathcal{H}^{\otimes N}$, it would still remain separable. Consequently, the state ϱ' would be separable, which is the contradiction. Thus, partial transposition of M_{2q} must be negative. Now, since M_{2q} is formed by all nonzero elements of ϱ', we maintain that also the state ϱ' must violate the Peres criterion, i.e., ϱ'^{T_B} must have a negative eigenvalue. Now, let ψ be the eigenvector corresponding to the eigenvalue. As the vector belongs to the subspace $\mathcal{H}_{2q}$ it follows that the matrix elements $\langle\psi|\varrho'^{T_B}|\psi\rangle$ and $\langle\psi|(\varrho^{\otimes N})^{T_B}|\psi\rangle$ are equal. Hence we obtain

$$\langle\psi|(\varrho^{\otimes N})^{T_B}|\psi\rangle < 0\,. \tag{10}$$

Thus, the state $\varrho^{\otimes N}$ violates the partial transposition criterion. However, as was mentioned, this implies that ϱ also does. The above consideration can be formally summarized as follows. If the output state of Alice-Bob action appears to have negative partial transposition, then the basic component ϱ of input state $\varrho^{\otimes N}$ must have also had negative partial transposition. This means nothing but that any action of type (4) on ϱ (including collecting N pairs) preserves positivity of partial transposition. In fact, it can be proved in a simpler way [21] by using the observation that $(A \otimes B\varrho C \otimes D)^{T_B} = A \otimes D^T \varrho^{T_B} C \otimes B^T$ for any operators A, B, C, D (here T stands for usual transposition). Then it immediately follows that action of *any* superoperator of type $\frac{1}{M}\sum_i A_i \otimes B_i \varrho^{\otimes N} A_i^\dagger \otimes B_i^\dagger$ producing an arbitrary two-component system (not necessarily a 2×2 one) preserves positivity of partial transposition.

Thus, we showed that, if a state ϱ is distillable, it must violate the Peres separability criterion. It is an important result as it implies that there are inseparable states which *cannot* be distilled. Indeed, quite recently one of us [19] constructed inseparable states which do not violate the criterion. Some of those peculiar states are density matrices for two spin-1 particles (the two-trit case). Using the standard basis for this case ($|1\rangle|1\rangle$, $|1\rangle|2\rangle$, $|1\rangle|3\rangle$, $|2\rangle|1\rangle$, $|2\rangle|2\rangle$, and so on), those matrices can be written in the following form:

$$\varrho_a = \frac{1}{8a+1} \times \begin{bmatrix} a & 0 & 0 & 0 & a & 0 & 0 & 0 & a \\ 0 & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a & 0 & 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 & a & 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 & 0 & a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1+a}{2} & 0 & \frac{\sqrt{1-a^2}}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & a & 0 \\ a & 0 & 0 & 0 & a & 0 & \frac{\sqrt{1-a^2}}{2} & 0 & \frac{1+a}{2} \end{bmatrix}, \tag{11}$$

with $0 < a < 1$. It has been shown [19] by means of independent separability criterion that those states are *inseparable* despite the fact that they have *positive* partial transposition. However, as we have shown above, the density matrices with positive partial transposition *cannot* be distilled to the singlet form. Consequently, any state of the form (11) cannot be distilled.

It is remarkable that the question whether a state is distillable or not has been reduced to the question of whether there is a two-qubit entanglement in a collection of N pairs for some N. Thus, the latter condition is *the necessary and sufficient condition* for any given state to be distilled. Indeed, as shown above, if a state ϱ is distillable then there exist two-dimensional projections P_A and P_B so that the state ϱ' given by Eq. (9) is inseparable. Conversely, if the latter condition is satisfied then ϱ can be distilled by projecting $\varrho^{\otimes N}$ locally by means of P_A and P_B and then applying the protocol proposed in [13] which is able to distill any two-qubit inseparable state. There is an open question as to whether the condition is equivalent to violation of the Peres criterion. Then, the latter would acquire the physical sense: Its violation would be equivalent to distillability.

Let us now discuss briefly the possible physical meaning of our result. As a matter of fact, we have revealed a kind of entanglement which cannot be used for sending reliably quantum information via teleportation. Using an analogy with thermodynamics [22], we can consider entanglement as a counterpart of energy, and sending of quantum information as a kind of *informational work*. Consequently, we can consider *free entanglement* (E_{free}) which can be distilled, and *bound entanglement* (E_{bound}). In particular, the free entanglement is naturally identified with distillable entanglement D as the latter asks us how many qubits can we reliably teleport via the mixed state. This kind of entanglement can always be converted via distillation protocol to the "active" singlet form.

To complete the analogy, one could consider the asymptotic number of singlets which are needed to produce a given mixed state as *internal entanglement* E_{int} (the counterpart of internal energy) [23]. Then, the bound entanglement can be quantitatively defined by the following equation:

$$E_{\text{int}} = E_{\text{free}} + E_{\text{bound}}\,. \tag{12}$$

In particular, for pure states we have $E_{\text{int}} = E_{\text{free}}$ and $E_{\text{bound}} = 0$. Indeed, pure states can be converted in a "lossless" way into active singlet form [18]. In the present Letter we show that there exist *inseparable* states having reciprocal properties. Namely, for the states of type (11), we have $E_{\text{int}} = E_{\text{bound}}$ and $E_{\text{free}} = 0$. Now, there are two possibilities: $E_{\text{int}} = 0$ or $E_{\text{int}} \neq 0$. Both cases are curious. In the first case, we would have *inseparable* states which can be produced from an asymptotically *zero* number of singlet pairs. This would imply, in turn, that entanglement of formation is *not* an additive state function [24], as by definition it does not vanish for any inseparable states. In the second case, we would have curious states which absorb entanglement in an irreversible way. To produce such states, one needs some amount of entanglement. But once the states were produced, there is no way to recover any, however little, piece of the initial entanglement. The latter is *entirely* lost.

A natural problem which arises in the context of the presented result is, What is the physical reason for which the partial transposition is connected with distillability? Our conjecture is that it is *time* which links intimately the two things. Indeed, transposition can be interpreted as the operation of time reversal [25] which has been recently considered in the context of the Peres criterion [26]. Also in the context of distillation, there appeared the problem of time. Namely, distillation is inherently connected with the quantum error correction for the quantum noisy channel supplemented by the two-way classical channel [7]. The quantum capacity of such channels can be strictly larger than without the classical channel. However, the price we must pay is that the error correction with two-way classical communication cannot be used to store the quantum information in noisy environment [7] because one cannot send the signal backward in time. Needless to say, deeper investigation of the connection among the distillation, partial transposition, and time reversal seems to be more than desirable.

Finally, it is perhaps worth mentioning the story of nonlocality of mixed states, beginning with the work of Werner [10]. He suggested that there are curious inseparable states which do not exhibit nonlocal correlations. Then, Popescu [27] showed that there is a subtle kind of pure quantum correlations which is exhibited by Werner mixtures. The distillability of all two-qubit states [13] proved that all of them are also nonlocal. One could suspect that the story will end by showing that all inseparable states can be distilled, hence they are nonlocal. Here, we showed that it is not true. So, one is now faced with the problem similar to the initial one, i.e., are the inseparable states with positive partial transposition nonlocal? Now, in view of the above result, it follows that the problem certainly cannot be solved by means of the distillation concept.

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