Space-Charge Dominated Beams in Synchrotrons

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The equations of motion for space-charge dominated beams in synchrotron are derived. We find that the space-charge force generates an identical defocusing function to the betatron motion and dispersion function. The self-consistent envelope equation obeys the Kapchinskij–Vladimirskij-type equation similar to that of the linear transport system. We employ these results to analyze the stability of the crystalline beams, and discuss the implication on the high intensity proton driver for the neutron spallation sources. [S0031-9007(98)06214-0]

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Intense charge particle beams have many applications. Some of these applications are proton drivers for the neutron spallation sources, the energy amplifier, the secondary beam sources such as muons, pions, and kaons, and heavy ion beam drivers for fusion energy [1,2]. Thus the stability of the space-charge dominated beam is an important topic in beam physics. There are two methods commonly used in the study of the stability of space-charge dominated beams. The linearized Vlasov equation method studies the threshold of an equilibrium distribution perturbatively [3], while the particle-core model studies the core stability using the envelope equation, and the particle stability using Hill's equation [4]. The linearized Vlasov equation approach has been shown to provide accurate description of the threshold behavior of collective modes. On the other hand, the particle-core model has been successfully used to describe the halo formation using parametric nonlinear resonances. Numerical simulations have been found to agree well with both the linearized Vlasov equation theory and the particle-core model in linear transport systems.

In the past, space-charge dominated beams were studied mainly in linacs, where the beam energy is low, and thus the space-charge force is important [5,6]. Since the synchrotron can accumulate linac beams and attain a much higher line density, the space-charge force can be as important. So far, the stability of space-charge dominated beams has not been fully explored in synchrotrons due to the complication of the dispersion functions, where we rely solely on multiparticle simulations. Deriving an envelope equation for a space-charge dominated beam in synchrotron is therefore an important timely task because the synchrotron is considered as the prevailing scheme for the bunch compression in the neutron spallation sources. The task is to find a self-consistent distribution and the envelope equation for the space-charge dominated beams in synchrotrons. This paper attempts to solve the task and discuss the effect of the strong space-charge force on the lattice functions.

In the curvilinear Frenet-Serret coordinate system with $(\mathbf{x}, \mathbf{s}, \mathbf{z})$ as unit basis vectors, the particle motion can be described by the phase space coordinate $(x, p_x, z, p_z, t, -E)$ [7], where the synchrotron phase space coordinates (t, E) are the time and the energy of the particle, x, z are betatron coordinates, and p_x, p_z are the corresponding conjugate coordinates. We consider only transverse force, where we choose $A_x = A_z = 0$ for the vector potential. The Hamiltonian up to the second order in p_x, p_z is

$$\tilde{H} \approx -\left(1 + \frac{x}{\rho}\right)p + \left(1 + \frac{x}{\rho}\right)\left(\frac{p_x^2 + p_z^2}{2p}\right) - eA_s\,, \tag{1}$$

where e and p are the charge and the momentum of the particle, ρ is the radius of curvature, and $A_s = (1 + x/\rho)\mathbf{A}\cdot\mathbf{s}$ is a component of the vector potential with $B_z = [1/(1 + x/\rho)](\partial A_s/\partial x)$, $B_x = -[1/(1 + x/\rho)](\partial A_s/\partial z)$. We expand the momentum about the reference value p_0 and obtain

$$\Delta p = p - p_0 \approx \frac{\Delta E}{\beta_0 c} - \frac{1}{2p_0}\left(\frac{\Delta E}{\beta_0 c \gamma_0}\right)^2 - \frac{eV_{\rm sc}}{\beta_0 c}\,, \tag{2}$$

where $\Delta E = E - E_0$ is the energy deviation, and β_0 and γ_0 are the Lorentz factors of the reference particle. Here $V_{\rm sc}$ is the self-consistent space-charge scalar potential that satisfies the Poisson equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right)V_{\rm sc} = -\frac{e}{\epsilon_0}\iiint dp_x\,dp_z \times d(-\Delta E) f(x, z, p_x, p_z, -\Delta E; s)\,, \tag{3}$$

where the distribution function f obeys the Vlasov equation $df/ds = 0$. The equilibrium distribution function in a synchrotron must also be a periodic function of s, and is generally a function of the effective Hamiltonian that includes the space-charge mean field potential. For a few special distributions, self-consistent space-charge potential can be expressed in analytic form, e.g., the Kapchinskij-Vladimirskij (KV) distribution [8] in a linear transport system.

Now we consider the case of a coasting beam without synchrotron motion so that the longitudinal electric field is zero. The vector potential is given by

$$A_s = B_0 x + \frac{B_0}{2\rho} x^2 + \frac{1}{2} B_1 (x^2 - z^2) + A_{sc}, \quad (4)$$

where $B_1 = \partial B_z / \partial x$ is the focusing function evaluated at the reference orbit, and A_{sc} is the vector potential due to the space-charge force. Since the magnetic force is equal to $-\beta_0^2$ times the electric force, we obtain $A_{sc} = \beta_0^2 V_{sc}/\beta_0 c$. Substituting the vector potential into the Hamiltonian, one obtains

$$H_0 = -p_0 \frac{\Delta E}{\beta_0^2 E_0} + p_0 \frac{1}{2\gamma_0^2}\left(\frac{\Delta E}{\beta_0^2 E_0}\right)^2 - p_0 \frac{\Delta E}{\beta_0^2 E_0} \frac{x}{\rho} + \frac{p_x^2 + p_z^2}{2p_0} + \frac{p_0}{2}(K_x x^2 + K_z z^2) + \frac{e}{\beta_0 c \gamma_0^2} V_{sc}, \quad (5)$$

where $K_x = 1/\rho^2 - B_1/B\rho$ and $K_z = B_1/B\rho$ are the focusing functions for betatron motion, $B\rho = |p_0/e|$ is the magnetic rigidity of the beam, and $B_0 = -p_0/e\rho$ signifies the expansion of x around the reference orbit.

The next step is to transform the coordinate system onto the closed orbit for the particle at energy E. Using the generating function [9]

$$F_2(x, \overline{p}_x, , z, \overline{p}_z, t, -\overline{\Delta E}) = \left(x - D_x \frac{\overline{\Delta E}}{\beta_0^2 E_0}\right)\overline{p}_x + \left(z - D_z \frac{\overline{\Delta E}}{\beta_0^2 E_0}\right)\overline{p}_z - (E_0 + \overline{\Delta E})t + x \frac{D_x'}{\beta_0 c}\overline{\Delta E} - \frac{1}{2} D_x D_x' p_0 \left(\frac{\overline{\Delta E}}{\beta_0^2 E_0}\right)^2 + z \frac{D_z'}{\beta_0 c}\overline{\Delta E} - \frac{1}{2} D_z D_z' p_0 \left(\frac{\overline{\Delta E}}{\beta_0^2 E_0}\right)^2,$$

and making a scale change to coordinates $\tilde{p}_x = \overline{p}_x/p_0$, $\tilde{p}_z = \overline{p}_z/p_0$, and $W = \overline{\Delta E}/p_0$, the new Hamiltonian becomes

$$\tilde{H}_1 = -\frac{W}{\beta_0 c} + \frac{\tilde{p}_x^2}{2} + \frac{1}{2} K_x \tilde{x}^2 + \frac{\tilde{p}_z^2}{2} + \frac{1}{2} K_z \tilde{z}^2 + \frac{W}{\beta_0 c}\tilde{x}\left(D_x'' + K_x D_x - \frac{1}{\rho}\right) + \frac{W}{\beta_0 c}\tilde{z}(D_z'' + K_z D_z) + \frac{1}{2}\left(\frac{W}{\beta_0 c}\right)^2 \left[\frac{1}{\gamma_0^2} + D_x\left(D_x'' + K_x D_x - \frac{2}{\rho}\right) + D_z(D_z'' + K_z D_z)\right] + \frac{e}{\beta_0 c p_0 \gamma_0^2} V_{sc}.$$

Note that $\tilde{x}$ is the betatron coordinate around the off-momentum closed orbit. Since the equilibrium distribution is a function of x and z, the mean field Coulomb potential is given by

$$V_{sc} \approx V_{sc,0} + \frac{1}{2} V_{sc,xx}\left(\tilde{x} + D_x \frac{W}{\beta_0 c}\right)^2 + V_{sc,xz}\left(\tilde{x} + D_z \frac{W}{\beta_0 c}\right)\left(\tilde{z} + D_z \frac{W}{\beta_0 c}\right) + \frac{1}{2} V_{sc,zz}\left(\tilde{z} + D_z \frac{W}{\beta_0 c}\right)^2 + \ldots, \quad (6)$$

where $V_{sc,0}$ is a constant term, and

$$V_{sc,xx} = \frac{\partial^2 V_{sc}}{\partial \tilde{x}^2}, \qquad V_{sc,zz} = \frac{\partial^2 V_{sc}}{\partial \tilde{z}^2}, \qquad V_{sc,xz} = \frac{\partial^2 V_{sc}}{\partial \tilde{x}\partial \tilde{z}}$$

are partial derivatives of the space-charge potential evaluated at the reference orbit. Eliminating the cross terms in the Hamiltonian, the equations for the dispersion functions are given by

$$D_x'' + \left(K_x + \frac{eV_{sc,xx}}{\beta_0 c p_0 \gamma_0^2}\right) D_x + \frac{eV_{sc,xz}}{\beta_0 c p_0 \gamma_0^2} D_z = \frac{1}{\rho}, \quad (7)$$

$$D_z'' + \left(K_z + \frac{eV_{sc,zz}}{\beta_0 c p_0 \gamma_0^2}\right) D_z + \frac{eV_{sc,xz}}{\beta_0 c p_0 \gamma_0^2} D_x = 0. \quad (8)$$

Note that the space-charge mean field reduces the focusing strength and may introduce a linear coupling to the equations of motion. The new Hamiltonian becomes

$$\tilde{H}_2 = \frac{1}{2}\left[\tilde{p}_x^2 + \left(K_x + \frac{eV_{sc,xx}}{\beta_0 c p_0 \gamma_0^2}\right)\tilde{x}^2 + \tilde{p}_z^2 + \left(K_z + \frac{eV_{sc,zz}}{\beta_0 c p_0 \gamma_0^2}\right)\tilde{z}^2 + 2\frac{eV_{sc,xz}}{\beta_0 c p_0 \gamma_0^2}\tilde{x}\tilde{z}\right] - \frac{W}{\beta_0 c} + \frac{1}{2}\left(\frac{W}{\beta_0 c}\right)^2\left[\frac{1}{\gamma_0^2} - \frac{D_x}{\rho}\right].$$

The synchrotron equation of motion is given by

$$\frac{d(\Delta\tilde{t})}{ds} = \frac{1}{\beta_0 c}\left(\frac{D_x}{\rho} - \frac{1}{\gamma_0^2}\right)\frac{W}{\beta_0 c}, \quad (9)$$

$$\frac{d(-W)}{ds} = 0, \quad (10)$$

where $\Delta\tilde{t} = \tilde{t} - s/\beta_0 c$ is the relative time. For a coasting beam, W is constant. The corresponding momentum compaction factor for the space-charge dominated beam

$$\alpha_{c,sc} = \frac{1}{C}\oint \frac{D_x}{\rho} ds, \quad (11)$$

where C is the circumference of the synchrotron, has an identical form as that of the emittance dominated beams except that the dispersion function is modified by the space-charge mean field. The equations of betatron motion are given by

$$\tilde{x}'' + \left(K_x + \frac{eV_{\mathrm{sc},xx}}{\beta_0 c p_0 \gamma_0^2}\right)\tilde{x} + \frac{eV_{\mathrm{sc},xz}}{\beta_0 c p_0 \gamma_0^2}\,\tilde{z} = 0\,, \quad (12)$$

$$\tilde{z}'' + \left(K_z + \frac{eV_{\mathrm{sc},zz}}{\beta_0 c p_0 \gamma_0^2}\right)\tilde{z} + \frac{eV_{\mathrm{sc},xz}}{\beta_0 c p_0 \gamma_0^2}\,\tilde{x} = 0\,. \quad (13)$$

We observe that Hill's equations for betatron motion have an identical focusing function as that of the dispersion functions. Furthermore, the space-charge force may introduce linear coupling to betatron motion and gives rise to the vertical dispersion.

We now apply our formalism to analyze the crystalline beams in a storage ring as they are the ultimate form of space-charge dominated beams [10]. By using a properly tailored "tapered" cooling force, an ordered state of the crystal beam can be obtained by the molecular dynamics numerical simulations [11]. When the crystalline state is formed, the normalized temperatures, defined in Ref. [11], will be less than 10^{-4} in all degrees of freedom. Figure 1 shows the dispersion function x_{co}/δ, and the vertical closed orbit $z_{\mathrm{co}}/\langle z_{\mathrm{co}}\rangle$ in one period of the lattice. The existence of an unique dispersion function shown in the upper curve of Fig. 1 indicates that the horizontal closed orbit of each particle is related to the dispersion function. The fact that $z_{\mathrm{co}}/\langle z_{\mathrm{co}}\rangle \approx 1$ for all particles in the crystalline beam indicates that (1) the space-charge force has almost fully compensated the quadrupole focusing force, and (2) there is no vertical dispersion function and no inhomogeneous term in Eq. (13). Thus the linear coupling due to the space-charge force is small, i.e., $V_{\mathrm{sc},xz} \approx 0$.

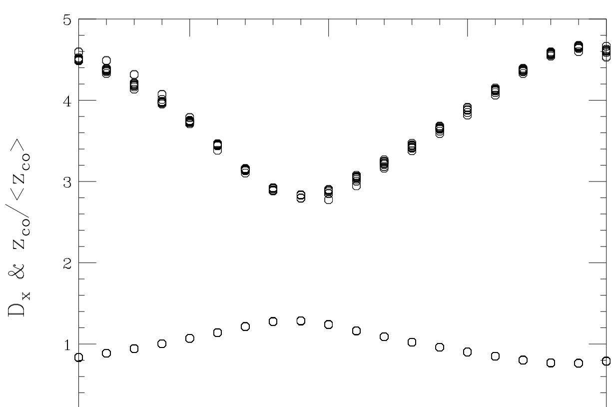

FIG. 1. The horizontal orbit x_{co}/δ (normalized unit) and the vertical orbit $z_{\mathrm{co}}/\langle z_{\mathrm{co}}\rangle$ of 20 particles obtained from the molecular dynamics calculations are plotted in one superperiod. The lattices used are simple periodic focusing and defocusing (FODO) lattice with superperiod $P = 10$ and betatron tunes $Q_x = Q_z = 2.34$. It is evident that the horizontal closed orbit of the crystalline particle arises from dispersion function, and the vertical closed orbit is obtained from the homogeneous Hill's equation of Eq. (13) with a focusing force almost fully balanced by the Coulomb mean field.

Neglecting the linear coupling and nonlinear contribution from the space-charge potential, the betatron Hamiltonian becomes

$$\tilde{H}_3 = \frac{1}{2}\left[\tilde{p}_x^2 + \tilde{p}_x^2 + \left(K_x + \frac{eV_{\mathrm{sc},xx}}{\beta_0 c p_0 \gamma_0^2}\right)\tilde{x}^2 + \left(K_z + \frac{eV_{\mathrm{sc},zz}}{\beta_0 c p_0 \gamma_0^2}\right)\tilde{z}^2\right], \quad (14)$$

and the KV distribution is a self-consistent solution [8]. In the KV model, we obtain

$$V_{\mathrm{sc},xx} = -\frac{eN}{\pi\epsilon_0}\frac{1}{a(a+b)}, \qquad V_{\mathrm{sc},zz} = -\frac{eN}{\pi\epsilon_0}\frac{1}{b(a+b)}, \quad (15)$$

where N is the number of particles per unit length, and a and b are the horizontal and the vertical betatron beam envelopes. For a KV beam, the envelope equations become

$$a'' + K_x a - \frac{\epsilon_x^2}{a^3} = \frac{2K_{\mathrm{sc}}}{a+b}\,, \quad (16)$$

$$b'' + K_z b - \frac{\epsilon_z^2}{b^3} = \frac{2K_{\mathrm{sc}}}{a+b}\,, \quad (17)$$

where ϵ_x and ϵ_z are the beam emittances, $K_{\mathrm{sc}} = 2Nr_0/\beta_0^2\gamma_0^3$ is the space-charge perveance, and r_0 is the classical radius.

When the horizontal and vertical betatron tunes for the crystalline beam lattice are equal, the envelope equation for $b = a$, in the smooth approximation, can be written as [12]

$$a'' + \left(\frac{2\pi\nu}{C}\right)^2 a - \frac{\epsilon^2}{a^3} = \frac{K_{\mathrm{sc}}}{a}\,, \quad (18)$$

where ν is the betatron tune, and ϵ is the average emittance of the crystalline beam. The envelope tune is given by

$$\nu_{\mathrm{env}} = 2\nu\left[1 - \frac{\kappa}{\sqrt{1+\kappa^2}+\kappa}\right]^{1/2}, \quad (19)$$

where κ is the normalized space-charge parameter given by $\kappa = K_{\mathrm{sc}}C/4\pi\nu\epsilon$. Here we note that the envelope tune is 2ν for an emittance dominated (hot) beam and $\sqrt{2}\nu$ for a space charge dominated (cold) beam. If the envelope tune encounters a systematic half-integer stop band (Mathieu instability), the envelope (or betatron amplitude function) will be unstable.

For a synchrotron made of P superperiods, systematic half-integer stop bands occur at $P/2, P, 3P/2, \ldots$. Therefore, in order to maintain a crystalline beam, the lattice must satisfy $\sqrt{2}\nu \le P/2$ [11].

Figure 2 shows the beam temperature, obtained from the molecular dynamics calculations, vs $\sqrt{2}\nu/P$. When the betatron tune of a cold beam reaches the envelope stop band at $\sqrt{2}\nu/P = 1/2$, the betatron envelope for all off-momentum particles becomes unstable, and the temperature of the beam increases suddenly. When the betatron

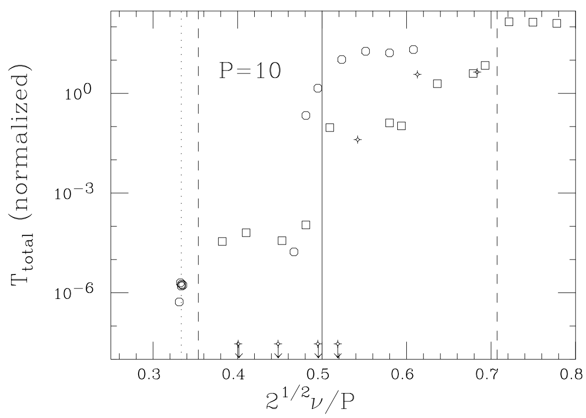

FIG. 2. The total temperature $T_{\text{total}} = T_x + T_z + T_s$ (in the normalized unit [11]) obtained from the molecular dynamics (MD) simulations with several different lattices are plotted as a function of the betatron tune. The circles and squares are obtained from lattice made of FODO cells with 10 superperiods, and the stars are obtained from the lattice with 6 superperiods of the TARN-11 storage ring at Institute of Nuclear Studies in Tokyo. The arrow attached to the star symbol shows that the beam temperature obtained from MD simulations is lower than 10^{-8}. Note that the solid line at $\sqrt{2}\,\nu = P/2$ corresponds to the envelope stop band of a cold beam and dashed lines at $2\nu = P/2$ and P correspond to that of a hot beam. The dotted line shows the corresponding third order envelope stop band of a cold beam.

tune of a hot beam encounters the envelope stop band at $2\nu/P = 1/2, 1$ shown as dashed lines in Fig. 2, the beam temperature seems to show a small stepwise increase as well. Besides the linear Mathieu instability, nonlinear systematic stop bands occur at $\sqrt{2}\,\nu = P/m\,(m = 3, 4, \ldots)$ [4]. Fortunately, higher order stop bands have zero width, and these stop bands can be easily suppressed by the cooling force.

For high density beams in synchrotrons, the space-charge parameter is designed to satisfy the condition $\nu\kappa \leq 0.4$, which is small in comparison with that of the linac beams or the crystalline beams. However, the beam particles stay in the synchrotron for a long time, and the accumulated effect can be as important. Here the systematic and random half-integer stop band for the envelope equation may play an essential role in the stability of the space-charge dominated beams. Comparison of numerical simulations with the theory presented in this Letter would be valuable.

In conclusion, we have derived the equations of motion for space-charge dominated beams in synchrotrons. We find that the space-charge defocusing field on the betatron coordinate $\tilde{x}$ and dispersion function D_x are identical. The momentum compaction factor of a synchrotron with space-charge dominated beams can be calculated by Eq. (11) with the modified dispersion function. We find that the KV beam is also a self-consistent distribution for the space-charge dominated beams in synchrotrons. For a KV beam, the envelope equations of motion are identical to that of linear transport channels. Our theory is found to be consistent with the numerical results obtained from a molecular dynamics simulation for the crystalline beam. Our formalism can be extended readily to solve the space-charge dominated bunched beam problems.

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[12] As the density of a space-charge dominated beam becomes higher, the intrabeam scattering becomes more important, and the heating rate increases. However, when a crystalline structure is formed, the random scattering vanishes. The crystalline state corresponds to a state with vanishing small transverse and longitudinal emittances.