Theory of Incoherent Dark Solitons

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We formulate the theory of incoherent dark spatial solitons in noninstantaneous self-defocusing nonlinear media. We find that the basic modal constituents of these incoherent dark soliton entities are radiation modes as well as bound states. Our results explain for the first time why incoherent dark solitons are in fact gray and why a transverse π-phase flip can facilitate their observation. [S0031-9007(98)06257-7]

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Until quite recently, the commonly held impression was that optical solitons are inherently coherent structures. Lately, however, two experimental studies have demonstrated beyond doubt that incoherent spatial solitons are also possible [1]. More specifically, incoherent bright solitons were found to exist in noninstantaneous nonlinear media such as biased photorefractives. In order to explain these newly observed entities, two complementary theoretical methods have been developed [2,3]. The first is the so-called coherent density approach which is by nature better suited to analyze the behavior and coherence properties of incoherent beams under dynamical propagation conditions [2]. The second method is a self-consistent modal theory, which is capable of identifying stationary incoherent bright solitons, their range of existence, and their coherence properties [3]. The equivalence of these two methods was later established in saturable nonlinear media of the logarithmic type where both approaches were found to exhibit analytical results [4]. Another approach based on geometrical optics has been recently suggested in the limit of broad incoherent beams [5]. In view of these developments, one may now pose the following important question: Are incoherent dark solitons also possible in nonlinear media? To resolve this issue, a computational study was recently undertaken using the coherent density method [6]. The numerical results of this work suggested that incoherent dark quasisolitons can be effectively excited in self-defocusing (reverse biased) photorefractive crystals, provided that, at the origin, a π-phase jump is imposed on the incoherent wave front [6]. Even more importantly, unlike their coherent counterparts [7], these dark incoherent solitons were always found to be gray! The gray character of these solutions is in qualitative agreement with some earlier predictions of random-phase envelope solitons made by Hasegawa two decades ago within the context of plasma physics [8]. In that early pioneering work, the average dynamics of all of the random quasiparticles involved were treated using a Vlasov transport equation.

Last month, incoherent dark-stripe and dark-hole (vortex) solitons were experimentally demonstrated in a biased photorefractive crystal [9]. In all cases, these self-trapped incoherent beams carried the characteristic signature of dark incoherent soliton structures [6,8]; i.e., they were found to be gray. Moreover, in agreement with predictions [6], the incoherent dark solitons were experimentally observed when an appropriate phase profile was imposed on the wave front. Yet, at this point, several important questions remain unanswered. First of all, are there truly stationary incoherent dark solitons and why are they gray? Furthermore, why is the π-phase jump necessary for their excitation and how is it possible for this initial phase imprint to survive in the midst of random-phase fluctuations? The answers to the above questions cannot be obtained from the coherent density method (because of its inherent complexity) nor from the approximate Vlasov approach. These issues can be resolved only by identifying the modal composition of these dark incoherent soliton states, as was done in the case of their bright soliton counterparts [3].

In this Letter, by means of an exact solution, we demonstrate for the first time that stationary incoherent dark solitons can exist in noninstantaneous nonlinear self-defocusing media. These solitons involve, in general, a belt of radiation modes (both odd and even) as well as bound states. The presence of even radiation and bound modes explains why these structures are in fact gray. Moreover, we find that the odd radiation modes dominate within the dark region of the beam, which explains the π-phase shift required to excite these dark incoherent soliton states. The coherence properties of these solitons are also considered, and they are found to be in good agreement with the results of previous computational studies [6].

Let us consider a self-defocusing nonlinear medium of the Kerr type, i.e., $n^2 = n_0^2 - n_2 I$, where n_0 is the linear refractive index of the material, n_2 is the Kerr coefficient, and I is the optical intensity. We assume that the nonlinearity responds much slower than the

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characteristic phase fluctuation time across the beam so as to avoid beam breakup due to speckle instabilities [3,10]. Thus, in this regime, the material will experience only the time-averaged beam intensity. Such noninstantaneous Kerr-like media include biased photorefractives at low intensity ratios and materials with appreciable thermal nonlinearities [11]. For example, a typical phase fluctuation time is 1 μs, whereas a photorefractive crystal responds within 0.1 s [9]. Let the time-averaged intensity profile of this planar dark incoherent soliton be of the form

$$I_s = I_0[1 - \varepsilon^2 \operatorname{sech}^2(x/x_0)], \tag{1}$$

where the parameter $\varepsilon^2 \le 1$ is associated with its grayness and x_0 is related to its spatial extent. The partially spatially incoherent dark beam is quasimonochromatic, and it propagates along z. Furthermore, let the electric field of all of the modes comprising this beam be written as $E = U(x)\exp(i\beta z)$, where β is the mode propagation constant. Using Eq. (1), the modal function U is then found to obey the following Helmholtz equation:

$$\frac{d^2U}{ds^2} + [g + f \operatorname{sech}^2(s)]U = 0, \tag{2}$$

where $s = x/x_0$, $g = [k_0^2(n_0^2 - n_2 I_0) - \beta^2]x_0^2$, and $f = k_0^2 x_0^2 \varepsilon^2 n_2 I_0$. In the spirit of Ref. [3], the next task will be to identify an appropriate modal composition such that the time-averaged intensity I_s gives rise to a nonlinear index change which is self-consistent [12] with the composition assumed in the very beginning. In general, Eq. (2) exhibits two types of eigenfunctions: radiation modes and bound modes. As shown schematically in Fig. 1, bound states are possible whenever $g = -q^2$ or $\beta^2 > k_0^2(n_0^2 - n_2 I_0)$, whereas radiation modes require that $g = +Q^2$ or $\beta^2 < k_0^2(n_0^2 - n_2 I_0)$.

At this point, let us first assume that the waveguide induced by this dark beam can support only one bound mode. This latter requirement can be met provided that the coefficient of the $\operatorname{sech}^2(s)$ potential is set equal to two; i.e., $f = 2$ or $x_0^2 = 2/(k_0^2 \varepsilon^2 n_2 I_0)$ [13]. In this case, all possible modes allowed by Eq. (2) are given by [14]

$$U_b = \operatorname{sech}(s), \tag{3a}$$

$$U_{r,e} = Q\cos(Qs) - \tanh(s)\sin(Qs), \tag{3b}$$

$$U_{r,o} = Q\sin(Qs) + \tanh(s)\cos(Qs). \tag{3c}$$

For $f = 2$, U_b is the only allowed bound state (at $q^2 = 1$), and the two degenerate eigenfunctions $U_{r,e}$ and $U_{r,o}$ are part of the radiation-mode continuum. It is important to note that $U_{r,e}$ is an even radiation mode whereas $U_{r,o}$ is odd. Following these results, the total electric field is given by [13]

$$E = c_b U_b(s)\exp(i\beta_b z) + \int_0^\infty dQ[\tilde{c}_e(Q)U_{r,e}(s,Q) + \tilde{c}_o(Q)U_{r,o}(s,Q)]\exp[i\beta_r(Q)z], \tag{4}$$

where c_b and $\tilde{c}_{e,o}$ are modal field coefficients that in general vary randomly in time [3,4]. In Eq. (4), the upper limit of the integral is taken at infinity. As we will see, Q is typically in the neighborhood of $Q \approx 0$, far away from $Q_{\max} = k_0(n_0^2 - n_2 I_0)^{1/2} x_0$, which in turn sets the upper limit of this integral. Under incoherent excitation, the following relationships hold true: $\langle c_{b,m} c_{b,n}^* \rangle \propto \delta_{mn}$, $\langle c_b \tilde{c}_{e,o}^* \rangle = 0$, $\langle \tilde{c}_o \tilde{c}_e^* \rangle = 0$, and $\langle \tilde{c}_e(Q)\tilde{c}_e^*(Q') \rangle = \langle \tilde{c}_o(Q)\tilde{c}_o^*(Q') \rangle \propto D(Q)\delta(Q - Q')$. In other words, the statistical time expectation value of the c-field coefficients is zero between different bound modes, and the same applies between bound modes and radiation modes (odd or even). Furthermore, the odd and even radiation modes are always uncorrelated. Among the even radiation modes, the $\tilde{c}_e$ coefficients correlate only for the same value of Q, and this is also true for the odd radiation fields. The last relationship also implies that the odd and even radiation modes are equally excited at the same Q [with strength $D(Q)$]. This is because the random source shows no preference to either odd or even radiation modes. The positive function $D(Q)$ represents a radiation-mode distribution.

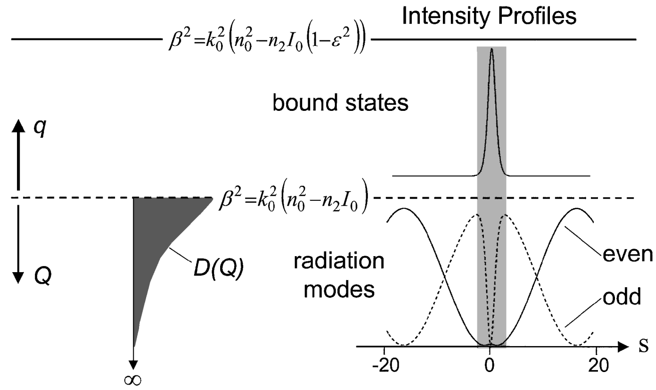

FIG. 1. Eigenvalue diagram associated with a first-order incoherent dark soliton. The bound state intensity as well as the intensities of the even and odd radiation modes (at $Q = 0.1$) are also depicted. The dark stripe on the right shows the spatial extent of the soliton-induced waveguide when $\varepsilon^2 = 0.5$.

By utilizing these latter relationships, the intensity $I \propto \langle E(s,z)E^*(s,z) \rangle$ can then be obtained from Eq. (4), i.e.,

$$I = A^2 \operatorname{sech}^2(s) + \int_0^\infty D(Q)[Q^2 + \tanh^2(s)]\,dQ, \tag{5}$$

where in Eq. (5) we made use of the fact that $|U_{r,e}|^2 + |U_{r,o}|^2 = Q^2 + \tanh^2(s)$, and $\langle |c_b|^2 \rangle \propto A^2$. The first term in Eq. (5) arises from the bound mode whereas the second one from the combined intensity of odd and even radiation modes. For self-consistency, it is required that the intensity given by Eq. (5) is identical to I_s of Eq. (1). This is satisfied provided that

$$I_0 = \int_0^\infty D(Q)(Q^2 + 1)\,dQ, \tag{6a}$$

$$A^2 = \int_0^\infty D(Q)[1 - \varepsilon^2(Q^2 + 1)]\,dQ. \tag{6b}$$

The analytical solution given by Eqs. (6) clearly demonstrates that stationary incoherent dark solitons indeed exist. This is the first time we know of that a soliton was found to involve a *continuum* of radiation modes as well as bound states. Even more importantly, this new class of solitons is gray because of the presence of even radiation and bound modes. It is evident from Eqs. (6) that the radiation mode distribution function $D(Q)$ is by no means unique. In fact, infinitely many self-consistent solutions can be obtained, depending on the particular choice of $D(Q)$.

To further illustrate our results, let $D(Q)$ be Boltzmann-like, i.e., $D(Q) = D_0 \exp(-Q/Q_0)$, where Q_0 represents the Q width of this distribution. The exponentially decreasing character of $D(Q)$ can be justified whenever the angular power spectrum of the incoherent source decreases with the launch angle [2,6]. As a result, more power is expected to be coupled into small-angle ($Q \approx 0$) radiation modes than in those at higher Q's. For this specific choice of $D(Q)$, one quickly finds that $D_0 = (I_0/Q_0)(2Q_0^2 + 1)^{-1}$ and that $A^2 = I_0[1 - \varepsilon^2(2Q_0^2 + 1)](2Q_0^2 + 1)^{-1}$. Thus, $I_s = I_b + I_r$, where

$$I_b = \frac{I_0}{2Q_0^2 + 1}[1 - \varepsilon^2(2Q_0^2 + 1)]\,\mathrm{sech}^2(s)\,, \tag{7a}$$

$$I_r = \frac{I_0}{2Q_0^2 + 1}[2Q_0^2 + 1 - \mathrm{sech}^2(s)]\,. \tag{7b}$$

In Eqs. (7), I_b is the bound-mode intensity component of the dark incoherent soliton, and I_r is the intensity profile of the radiation-mode belt. It is also clear from Eq. (7a) that this soliton exists provided $\varepsilon^2(2Q_0^2 + 1) \leq 1$. The complex coherence factor $\mu_{1,2}(s_1, s_2)$ [15] of this incoherent soliton can then be obtained from Eq. (4) by evaluating the quantity

$$\begin{aligned}\langle E(s_1,z)E^*(s_2,z)\rangle \propto{}& A^2\,\mathrm{sech}(s_1)\,\mathrm{sech}(s_2)\\ &+ \int_0^\infty D(Q)\,[U_{r,e}(s_1)U_{r,e}(s_2)\\ &+ U_{r,o}(s_1)U_{r,o}(s_2)]\,dQ\,. \end{aligned}\tag{8}$$

In turn, its correlation length can be found from $l_c(s) = x_0 \int_{-\infty}^{\infty} |\mu_{1,2}(s, s + \delta)|^2\, d\delta$ [6].

Let us now physically interpret these results. From the $f = 2$ condition, one can deduce that, for a given $n_2 I_0$, the width x_0 of the dark soliton increases with its grayness. Moreover, it is important to note that Q_0 defines the correlation length at the tails ($s \to \pm\infty$) of this dark incoherent soliton. In these regions, the bound states disappear and the soliton correlation length is determined by the width Q_0 of the radiation-mode belt. At the tails, l_c decreases as Q_0 increases and vice versa. In fact, l_c in these regions coincides with the correlation length of the source. In the limit $Q_0 = 0$, $\varepsilon^2 = 1$, Eqs. (7) reduce to the well-known coherent dark spatial soliton solution [7]. In this case, $I_b = 0$ and the soliton consists of an odd $\tanh(s)$ mode at cutoff with $l_c = \infty$ everywhere. On the other hand, for $Q_0 = 0$ and $\varepsilon^2 \neq 1$, we obtain an incoherently coupled dark-bright soliton pair, identical in nature to that previously considered in photorefractive crystals [16]. From the condition $\varepsilon^2(2Q_0^2 + 1) \leq 1$, it is also clear that the dark soliton becomes more gray as its incoherence increases. Another interesting possibility arises in the limit $\varepsilon^2(2Q_0^2 + 1) = 1$. In this case, the bound state is empty ($I_b = 0$) and thus the dark incoherent soliton consists of only radiation modes. As previously noted, the dark incoherent soliton is actually gray because of the presence of even radiation and even bound modes. To further illustrate these issues, let us consider a practical example. Let $n_0 = 2$, $\lambda_0 = 0.5\ \mu$m, and $n_2 I_0 = 10^{-3}$. Let the soliton grayness be 50% or $\varepsilon^2 = 0.5$. These parameters are in fact close to those previously considered in photorefractives [6,9]. In this case, $x_0 \simeq 5\ \mu$m, and this soliton exists for $Q_0 \leq 1/\sqrt{2}$. Figures 2(a) and 2(b) show the soliton intensity profile and correlation length when $Q_0 = 0.4$. The correlation length of the source $\approx 13.5\ \mu$m. The depression in l_c at $s \approx 0$ is due to the presence of the bound mode. Figures 2(c) and 2(d) provide the same information when $Q_0 = 0.7$. This corresponds to a source correlation length of $\approx 5.3\ \mu$m, and in this case the bound mode is almost absent. For this reason, l_c increases around the dark notch. Overall, the behavior of the I_s and l_c curves is in qualitative agreement with the findings of previous studies [6]. From the above results, it becomes apparent that, for our choice of $D(Q)$, the radiation modes are mostly confined within a narrow belt around $Q \approx 0$. Because of this, the odd radiation modes dominate in the soliton-induced waveguide as shown schematically in Fig. 1. This behavior can be easily understood by considering Eqs. (3b) and (3c) in the neighborhood of $s \approx 0$ when $Q \approx 0$. Thus, in order to effectively launch this dark incoherent soliton the phase must be properly manipulated so that, at the center ($s = 0$), the field distribution is mostly odd. This explains

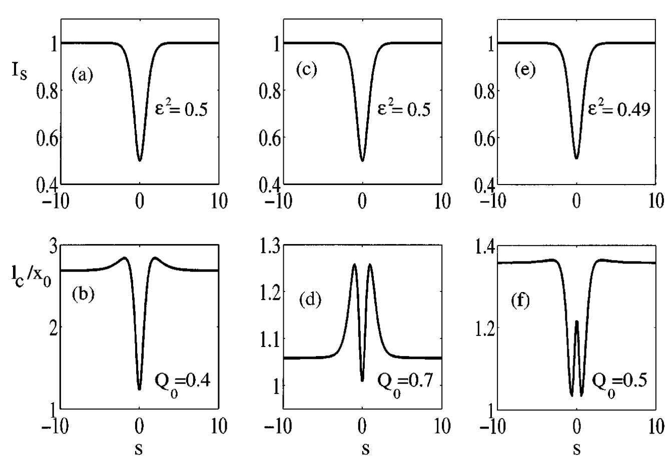

FIG. 2. Intensity profiles and corresponding correlation length curves of a first-order incoherent dark soliton when (a),(b) $\varepsilon^2 = 0.5$ and $Q_0 = 0.4$; (c),(d) $\varepsilon^2 = 0.5$ and $Q_0 = 0.7$. (e),(f) The same information for a second-order dark incoherent soliton when $\varepsilon^2 = 0.49$ and $Q_0 = 0.5$.

why a π-phase shift can greatly facilitate their observation [6,9]. Depending on initial conditions, the even bound mode may subsequently appear as a result of evolution.

Similarly, higher-order dark incoherent solitons can be obtained for $f = 6, 12, 20, \ldots$ [13,14]. For example, if $f = 6$, the soliton-induced potential can support two bound states, i.e., $U'_{b1} = \mathrm{sech}^2(s)$ and $U'_{b2} = \mathrm{sech}(s)\tanh(s)$ at $q_1^2 = 4$ and $q_2^2 = 1$, respectively. In this case ($f = 6$), the radiation modes are given by

$$U'_{r,e} = [1 + Q^2 - 3\tanh^2(s)]\cos(Qs) - 3Q\tanh(s)\sin(Qs), \tag{9a}$$

$$U'_{r,o} = [1 + Q^2 - 3\tanh^2(s)]\sin(Qs) + 3Q\tanh(s)\cos(Qs). \tag{9b}$$

Thus $I'_b = B^2\,\mathrm{sech}^4(s) + C^2\,\mathrm{sech}^2(s)\tanh^2(s)$. As in the case of first-order incoherent dark solitons, $D(Q)$ is not unique. If we choose, however, $D(Q) = D'_0\exp(-Q/Q_0)$, we find that $D'_0 = (I_0/2Q_0)(12Q_0^4 + 5Q_0^2 + 2)^{-1}$, $B^2 = Q_0D'_0[(3 + 6Q_0^2) - 2\varepsilon^2(12Q_0^4 + 5Q_0^2 + 2)]$, and $C^2 = B^2 + 9Q_0D'_0$. The intensity component of the radiation belt is given by

$$I'_r = D'_0Q_0[2(12Q_0^4 + 5Q_0^2 + 2) + 9\,\mathrm{sech}^4(s) - (12 + 6Q_0^2)\,\mathrm{sech}^2(s)]. \tag{10}$$

Again, $I'_b + I'_r = I_s$, where I_s is given by Eq. (1). From the above, $C > B$, and this soliton is possible provided $B^2 \geq 0$. In the limit $B = 0$, the first (even) bound mode is empty, and the soliton involves only radiation modes and the next odd bound state. Furthermore, for a given degree of grayness and nonlinear index change, the $f = 6$ dark soliton is broader than the first-order one by a factor of $\sqrt{3}$. Figures 2(e) and 2(f) depict the intensity profile and correlation length of such a second-order incoherent dark soliton, under the same parameters used before when $Q_0 = 0.5$, $\varepsilon^2 = 0.49$, and $x_0 \simeq 8.8\ \mu$m. The source correlation length is 12 μm. As shown in Fig. 2(f), the correlation length curve now exhibits a richer substructure within the local l_c minimum around $s \approx 0$. This is due to the presence of the additional odd bound mode. Figures 2(b), 2(d), and 2(f) also suggest that what was found in the computational study of Ref. [6] was actually a dark incoherent soliton of the first-order type with $A \approx 0$. More specifically, the l_c curve of the dark quasisoliton of Ref. [6] was very similar to that of Fig. 2(d) and had no substructure around $s \approx 0$ which is characteristic of a higher-order dark soliton state. It is also very important to note that, in this latter case ($f = 6$), the narrow ($Q \approx 0$) radiation-mode belt is dominated by even radiation modes within the waveguide region. On the other hand, our analysis shows that this second-order dark incoherent soliton has a strong contribution from the second odd bound state which, by the way, is never empty. Such higher-order dark solitons can be launched by properly "engineering" the input beam in a manner similar to that of Ref. [17]. The modal composition of even higher-order dark incoherent solitons ($f = 12, 20, \ldots$) can be obtained in a similar fashion.

In conclusion, we have theoretically demonstrated the existence of dark incoherent spatial solitons in noninstantaneous self-defocusing nonlinear media. These new dark soliton entities were found to involve radiation modes as well as bound states. Our results explain for the first time why these solitons are gray and why a π-phase shift tends to facilitate their experimental observation. Finally, it will be of interest to explore the possibility of extending our results in two dimensions, i.e., in the description of two-dimensional dark incoherent solitons [9].

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