Regular Black Hole in General Relativity Coupled to Nonlinear Electrodynamics

Eloy Ayón-Beato* and Alberto García
Departamento de Física, Centro de Investigación y Estudios Avanzados del IPN, Apdo. Postal 14-740, 07000 México DF, Mexico
(Received 14 November 1997)

The first regular *exact* black hole solution in general relativity is presented. The source is a nonlinear electrodynamic field satisfying the weak energy condition, which in the limit of weak field becomes the Maxwell field. The solution corresponds to a charged black hole with $|q| \leq 2s_c m \approx 0.6m$, having the metric, the curvature invariants, and the electric field regular everywhere. [S0031-9007(98)06332-7]

PACS numbers: 04.20.Jb, 04.70.Bw

In general relativity the existence of singularities appears to be a property inherent to most of the physically relevant solutions of Einstein equations, in particular, to all known up-to-date black hole *exact* solutions [1]. The Penrose cosmic censorship conjecture states that these singularities must be dressed by event horizons; no causal connection could exist between the interior of a black hole with the exterior fields, thus pathologies occurring at the singular region would have no influence on the exterior region, and the physics outside would be well behaved (cf. [2] for a review on the recent status of this conjecture).

To avoid the black hole singularity problem, some regular *models* has been proposed [3–8]. All of them have been referred to as "Bardeen black holes" [9], since Bardeen was the first author producing a surprising regular black hole model [3]. No one of these models is an *exact* solution to Einstein equations; there are no known physical sources associated with any of them. The attempts to solve this problem have usually been addressed to the search of more general gravity theories. The best candidate today to produce singularity-free solutions, even at the classical level, due to its intrinsic nonlocality, is string theory [10]. There are examples in other contexts, for instance, in $N = 1$ supergravity domain wall solutions with horizons but no singularities have been found (cf. [11], and references therein), another example is given in exact conformal field theory [12].

We show in this Letter that in the framework of the standard general relativity one can find singularity-free solutions of the Einstein field equations coupled to a suitable nonlinear electrodynamics, which in the weak field approximation becomes the usual linear Maxwell theory. Previous efforts on this direction with nonlinear electrodynamics either have been totally unsuccessful or only partially solve the considered singularity problem [13–15]. We propose a new nonlinear electrodynamics which coupled to gravity actually produces a nonsingular *exact* black hole solution satisfying the weak energy condition.

The gravitational field of our solution is described by the metric

$$\begin{aligned} \boldsymbol{g} = &-\left(1 - \frac{2mr^2}{(r^2+q^2)^{3/2}} + \frac{q^2r^2}{(r^2+q^2)^2}\right)d\boldsymbol{t}^2 \\ &+ \left(1 - \frac{2mr^2}{(r^2+q^2)^{3/2}} + \frac{q^2r^2}{(r^2+q^2)^2}\right)^{-1} d\boldsymbol{r}^2 \\ &+ r^2 d\boldsymbol{\Omega}^2, \end{aligned} \tag{1}$$

while the associated electric field E is given by

$$E = qr^4\left(\frac{r^2 - 5q^2}{(r^2+q^2)^4} + \frac{15}{2}\frac{m}{(r^2+q^2)^{7/2}}\right). \tag{2}$$

Notice that this solution asymptotically behaves as the Reissner-Nordström solution, i.e.,

$$\begin{aligned} -g_{tt} &= 1 - 2m/r + q^2/r^2 + O(1/r^3), \\ E &= q/r^2 + O(1/r^3), \end{aligned}$$

thus the parameters m and q are related correspondingly with the mass and the electric charge. For a certain range of the mass and charge our metric (1) is a black hole, which in addition is regular everywhere. Accomplishing the substitutions $x = r/|q|$ and $s = |q|/2m$, we rewrite g_{tt} as

$$-g_{tt} = A(x,s) \equiv 1 - \frac{1}{s}\frac{x^2}{(1+x^2)^{3/2}} + \frac{x^2}{(1+x^2)^2}, \tag{3}$$

which, for any nonvanishing value of s, has a single minimum; cf. Fig. 1. There exists a single real critical value of x, x_c, and one of s, s_c, to be determined from $A(x_c, s_c) = 0$ and $\partial_x A(x_c, s_c) = 0$, namely

$$\begin{aligned} t^4 - \frac{t^3}{s} + t^2 + \frac{t}{s} - 1 &= 0, \\ \frac{t^3}{s} - 2t^2 - \frac{3t}{s} + 4 &= 0, \end{aligned}$$

where $t^2 \equiv x^2 + 1$. To solve these equations, one substitutes $s = t(t^2 - 3)/(2t^2 - 4)$ from the second equation into the first one arriving at $t^6 - 4t^4 + 2t^2 - 1 = 0$, which has only one real solution for t^2, thus the corresponding critical values are $s_c \approx 0.317$ and $x_c \approx 1.58$. For $s < s_c$ the quoted minimum is negative, for $s = s_c$ the minimum vanishes, and for $s > s_c$ the minimum is

 0031-9007/98/80(23)/5056(4)$15.00

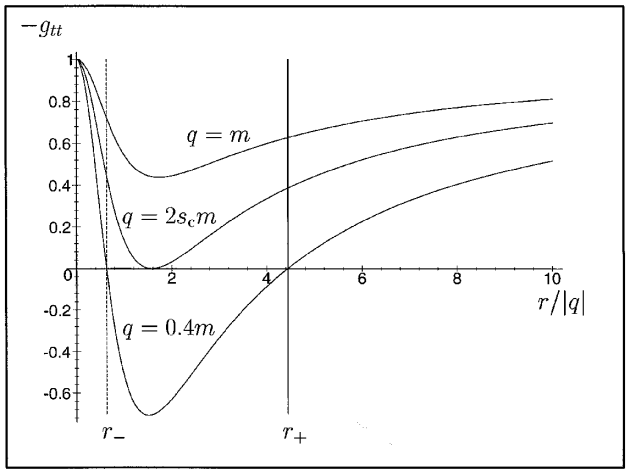

FIG. 1. Behavior of $-g_{tt}$ for different values of charge.

positive. Evaluating the curvature invariants R, $R_{\mu\nu}R^{\mu\nu}$, and $R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$ for metric (1) one establishes that they are all regular everywhere, cf. Fig. 2; thus for $s \leq s_c$ the singularities appearing in (1) due to the vanishing of A are only coordinate singularities describing the existence of horizons; consequently, we are in the presence of black hole solutions for $|q| \leq 2s_c m \approx 0.6m$.

For these values of mass and charge we have, under the strict inequality $|q| < 2s_c m$, inner and event horizons for the killing field $\boldsymbol{k} = \partial/\partial \boldsymbol{t}$, defined by the real solutions of the quartic equation $-k_\mu k^\mu = A = 0$, which are given by

$$r_\pm = |q|\left(\left[\frac{1}{4s} + \frac{\sqrt{f(s)}}{12s} \pm \frac{\sqrt{6}}{12s} \times \left(\frac{9}{2} - 12s^2 - \frac{f(s)}{6} - \frac{9(12s^2-1)}{\sqrt{f(s)}}\right)^{1/2}\right]^2 - 1\right)^{1/2}, \quad (4)$$

$$f(s) = 6\left(\frac{3}{2} - 4s^2 + s g(s)^{1/3} - \frac{4s(11s^2-3)}{g(s)^{1/3}}\right),$$

$$g(s) = 4(9s + 74s^3 + \sqrt{27(400s^6 - 112s^4 + 47s^2 - 4)})\,.$$

For $|q| = 2s_c m$, the horizons shrink into a single one, corresponding to an extreme black hole [$\nabla_\nu(k_\mu k^\mu) = 0$]. The extension of the metric beyond the horizons $r_\pm$ becomes apparent by passing to the standard advanced and retarded Eddington-Finkelstein coordinates, in terms of which the metric is smooth everywhere, even in the extreme case. Following step by step the procedure presented in Ref. [16] (Chap. V) to derive the global structure of the Reissner-Nordström black hole, one can arrive at the global structure of our solution and construct the Penrose diagrams; nevertheless, because of journal length restrictions we omit here the corresponding calculations and diagrams, leaving this issue for an extended publication. Briefly, what one encounters in the case of our nonextreme black hole solution, $|q| < 2s_c m$, is the splitting of the space-time into three regions, I: $r > r_+$, II: $r_- < r < r_+$, and III: $0 \leq r < r_-$; cf. Fig. 1. In each region one introduces advanced and retarded coordinates u and v, related with r through the so called tortoise coordinate $r^* \equiv \int A^{-1} dr$, which in our case is quite involved. Further, by the inversion of u and v, $u \to -u$, $v \to -v$, one obtains the remaining regions I′, II′, and III′. Introducing a new set of null coordinates, one arrives at the maximal extension of the nonextreme black hole. The Penrose diagram of the maximal analytical extension of our solution is obtained by gluing appropriately copies of these six regions upward and downward *ad infinitum.* In the extreme black hole case, $|q| = 2s_c m$, there arise two regions, I: $r > r_c$ and III: $0 \leq r < r_c$; cf. Fig. 1, in which again one introduces advanced and retarded u and v coordinates to accomplish the maximal analytical extension; these two regions determine the main building block of the extension. To construct the Penrose diagram of the maximal analytical extension, one glues copies of this block in a suitable way. In both cases, extreme and nonextreme, there is no singularity at $r = 0$, which is now simply the origin of the spherical coordinates. Summarizing, our space-time possesses the same global structure as the Reissner-Nordström black hole except that the singularity, at $r = 0$, of this last solution has been smoothed out.

For $|q| > 2s_c m$, there are no horizons and the corresponding exact solution represents a globally regular

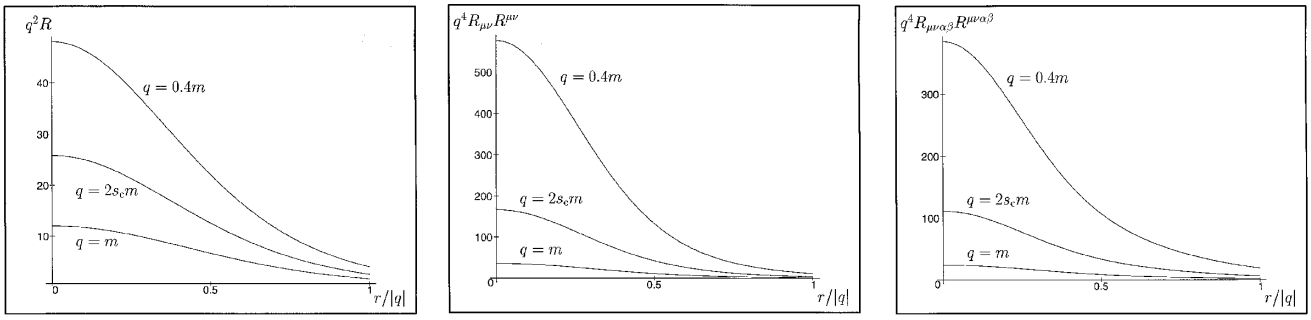

FIG. 2. Regular behavior of the Ricci, q^2R, Ricci square, $q^4 R_{\mu\nu}R^{\mu\nu}$, and the Riemann square, $q^4 R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$, scalars for different values of charge; the abscissa is $r/|q|$.

space-time. It is worthwhile to mention in this respect the existence of globally smooth solutions to the Einstein+matter (Yang-Mills, Yang-Mills-Higgs) equations; although there are demonstrations of the existence of these solutions [17,18], they are numerically given and there are no analytical closed expressions for them [19]; cf. [20], and references therein.

The fields (1) and (2) arise as a solution of the Einstein nonlinear electrodynamic field equations derived from the action proposed in Einstein-dual nonlinear electrodynamic theory [21], which in the studied case becomes

$$S = \int dv\left(\frac{1}{16\pi} R - \frac{1}{4\pi} \mathcal{L}(F)\right), \tag{5}$$

where R is scalar curvature and $\mathcal{L}$ is a function of $F \equiv \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$. Alternatively, one can describe the considered system using another function obtained by means of a Legendre transformation [21]:

$$\mathcal{H} \equiv 2F\mathcal{L}_F - \mathcal{L}\,. \tag{6}$$

Defining $P_{\mu\nu} \equiv \mathcal{L}_F F_{\mu\nu}$, it can be shown that $\mathcal{H}$ is a function of $P \equiv \frac{1}{4}P_{\mu\nu}P^{\mu\nu} = (\mathcal{L}_F)^2 F$, i.e., $d\mathcal{H} = (\mathcal{L}_F)^{-1} d[(\mathcal{L}_F)^2 F] = \mathcal{H}_P dP$. With the help of $\mathcal{H}$ one expresses the nonlinear electromagnetic Lagrangian in the action (5) as $\mathcal{L} = 2P\mathcal{H}_P - \mathcal{H}$, depending on the antisymmetric tensor $P_{\mu\nu}$. The specific function $\mathcal{H}$, determining the nonlinear electrodynamic source used, is given as

$$\mathcal{H}(P) = P\frac{\left(1 - 3\sqrt{-2q^2P}\right)}{(1 + \sqrt{-2q^2P})^3} - \frac{3}{2q^2 s}\left(\frac{\sqrt{-2q^2P}}{1 + \sqrt{-2q^2P}}\right)^{5/2}, \tag{7}$$

where $s = |q|/2m$ and the invariant P is a negative quantity. The corresponding Lagrangian occurs to be

$$\mathcal{L} = P\frac{\left(1 - 8\sqrt{-2q^2P} - 6q^2P\right)}{(1 + \sqrt{-2q^2P})^4} - \frac{3}{4q^2 s}\frac{(-2q^2P)^{5/4}\left(3 - 2\sqrt{-2q^2P}\right)}{\left(1 + \sqrt{-2q^2P}\right)^{7/2}}. \tag{8}$$

The function (7) satisfies the plausible conditions, needed for a nonlinear electromagnetic model, of (i) correspondence to Maxwell theory, i.e., $\mathcal{H} \approx P$ for weak fields ($P \ll 1$), and (ii) the weak energy condition, which requires $\mathcal{H} < 0$ and $\mathcal{H}_P > 0$; cf. Fig. 3. We would like to point out that our solution, in addition to being regular and to satisfying the weak energy condition, is characterized by another feature: it does not admit a Cauchy surface. Hence, it does not contradict the Penrose singularity theorem supported on the hypotheses of: fulfillment of the null energy condition, existence of a noncompact Cauchy surface, and existence of a closed trapped surface and concluding no null geodesically completeness of the space-time.

In what follows, we shall briefly give the main lines of the integration process yielding the studied solution. The Einstein and nonlinear electrodynamic equations arising from action (5) are

$$G^{\nu}_{\mu} = 2[\mathcal{H}_P P_{\mu\lambda}P^{\nu\lambda} - \delta^{\nu}_{\mu}(2P\mathcal{H}_P - \mathcal{H})], \tag{9}$$

$$\nabla_{\mu}P^{\alpha\mu} = 0\,. \tag{10}$$

In order to obtain the solution (1),(2), we consider the static and spherically symmetric configuration

$$\boldsymbol{g} = -\left(1 - \frac{2m}{r} + \frac{Q(r)}{r^2}\right)d\boldsymbol{t}^2 + \left(1 - \frac{2m}{r} + \frac{Q(r)}{r^2}\right)^{-1} d\boldsymbol{r}^2 + r^2 d\boldsymbol{\Omega}^2, \tag{11}$$

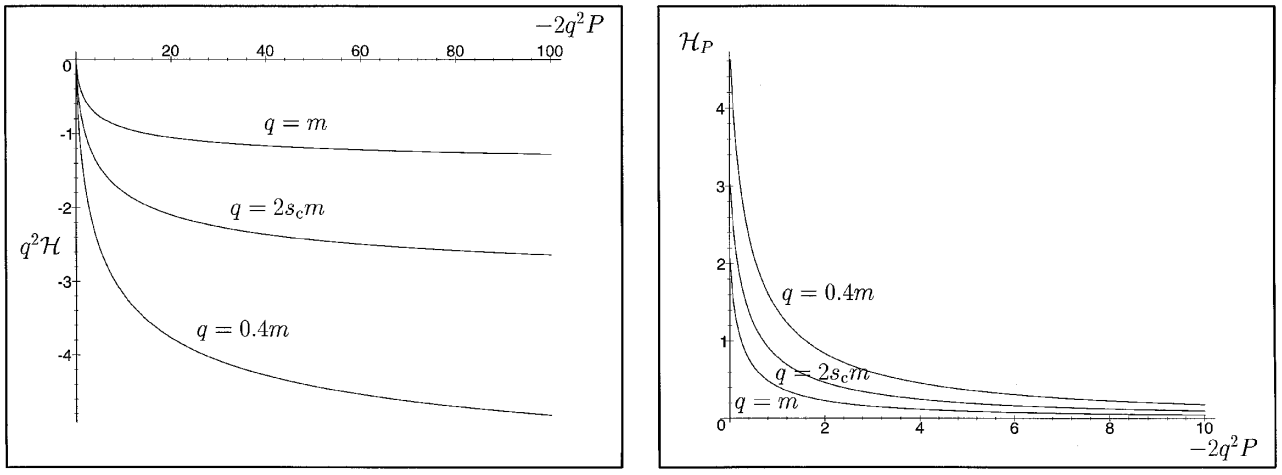

FIG. 3. Behavior of $q^2\mathcal{H}$ and $\mathcal{H}_P$ with respect to the positive abscissa $-2q^2P$ for different values of charge.

and the following ansatz for the antisymmetric field $P_{\mu\nu} = 2\delta^t_{[\mu}\delta^r_{\nu]}D(r)$. With these choices the equations (10) integrate as

$$P_{\mu\nu} = 2\delta^t_{[\mu}\delta^r_{\nu]}\frac{q}{r^2} \rightarrow P = -\frac{D^2}{2} = \frac{q^2}{2r^4}, \qquad (12)$$

where we have chosen the integration constant as q since, as it was previously anticipated, it actually plays the role of the electric charge. The evaluation of the electric field $E = F_{tr} = \mathcal{H}_P D$, using expression (7) for $\mathcal{H}$, gives just the formula (2). The ${}^t_t$ component of Einstein equations (9) yields the basic equation

$$\frac{rQ' - Q}{r^4} = 2\mathcal{H}(P). \qquad (13)$$

Substituting $\mathcal{H}$ from (7) with $P = -q^2/2r^4$ one can write the integral of (13) as

$$Q = q^2 r \int_r^\infty dy \left(\frac{6my^2}{(y^2 + q^2)^{5/2}} + \frac{y^2(y^2 - 3q^2)}{(y^2 + q^2)^3} \right), \qquad (14)$$

the integrand above can be expressed as $\partial_y[2my^3/q^2(y^2 + q^2)^{3/2} - y^3/(y^2 + q^2)^2]$, thus one arrives at

$$Q = 2mr - \frac{2mr^4}{(r^2 + q^2)^{3/2}} + \frac{q^2 r^4}{(r^2 + q^2)^2}. \qquad (15)$$

Substituting Q into $-g_{tt} = 1 - 2m/r + Q/r^2$ one finally gets Eq. (1).

This work was partially supported by the CONACyT Grant No. 3692P-E9607, and the Sistema Nacional de Investigadores (SNI). One of the authors E. A. B. thanks the staff of the Physics Department at CINVESTAV for support.

*On leave from Grupo de Física Teórica, CEMAFIT-ICIMAF, Calle E #309, esq. a 15, CP 10400, Ciudad Habana, Cuba.

[1] S. W. Hawking and G. F. Ellis, *The Large Scale Structure of Space-Time* (Cambridge University Press, Cambridge, England, 1973).
[2] R. M. Wald, gr-qc/9710068, 1997.
[3] J. Bardeen, in Proceedings of GR5, Tiflis, U.S.S.R., 1968.
[4] E. Ayón-Beato, graduate dissertation, Faculty of Physics, Havana University, 1993.
[5] A. Borde, Phys. Rev. D **50**, 3392 (1994).
[6] C. Barrabès and V. P. Frolov, Phys. Rev. D **53**, 3215 (1996).
[7] M. Mars, M. M. Martín-Prats, and J. M. M. Senovilla, Classical Quantum Gravity **13**, L51 (1996).
[8] A. Cabo and E. Ayón-Beato, gr-qc/9704073, 1997.
[9] A. Borde, Phys. Rev. D **55**, 7615 (1997).
[10] A. A. Tseytlin, Phys. Lett. B **363**, 223 (1995).
[11] M. Cvetic, Phys. Rev. Lett. **71**, 815 (1993).
[12] J. H. Horne and G. T. Horowitz, Nucl. Phys. **B368**, 444 (1992).
[13] H. P. Oliveira, Classical Quantum Gravity **11**, 1469 (1994).
[14] H. Soleng, Report No. CERN-TH/95-110, 1995.
[15] D. Palatnik, quant-ph/9701017, 1997.
[16] S. Chandrasekhar, *The Mathematical Theory of Black Holes* (Oxford University Press, New York, 1983).
[17] J. A. Smoller and A. G. Wasserman, Commun. Math. Phys. **151**, 303 (1993).
[18] P. Breitenlohner, P. Forgács, and D. Maison, Commun. Math. Phys. **163**, 141 (1994).
[19] R. Bartnik and J. McKinnon, Phys. Rev. Lett. **61**, 141 (1988).
[20] P. Bizoń, Acta Phys. Pol. B **25**, 877 (1994).
[21] H. Salazar, A. García, and J. Plebański, J. Math. Phys. **28**, 2171 (1987).