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# Searching for Higher Dimensional Integrable Models from Lower Ones via Painlevé Analysis

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Extending the Painlevé approach to a more general form, one can get infinitely many new integrable models under the meanings that they *possess conformal invariance and the Painlevé property* in any space dimensions from a given lower dimensional integrable model. Using the Kadomtsev-Petviashvili, nonlinear Schrödinger, and Schwarz Korteweg–de Vries equations as simple examples, some explicit (3 + 1)-dimensional integrable models are given. [S0031-9007(98)06288-7]

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Modern soliton theory is widely applied in almost all of the physics fields such as the field theory [1], condensed matter physics [2], fluid mechanics [3], plasma physics [4], optics [5], particle and nuclear physics [6], universe [7], geophysics [8], planetary and space science [9], and in other scientific and technological fields such as communications [10], chemistry [11], biology [12], etc. However, almost all of the known integrable models are only in (1 + 1) and (2 + 1) dimensions. Because the real physical space is (3 + 1) dimensional, various physicists and mathematicians have been trying to find some nontrivial (3 + 1)-dimensional integrable models [13,14], but there is little progress in this field. There is no one known real significant $(n + 1)$-dimensional ($n \geq 3$) integrable model except the "(2 + 2)"-dimensional self-dual Yang-Mills field equation [13].

To reduce a nonlinear partial differential equation (PDE) to some ordinary differential equations (ODEs) by using the classical and nonclassical Lie approaches [15–17] is one of the most effective methods for solving a nonlinear PDE. From the studies of the (1 + 1)- and (2 + 1)-dimensional integrable models, one knows that there are several ODE reductions, such as the Riccati equation and the Painlevé I–VI equations [18] for all known integrable (1 + 1)- and (2 + 1)-dimensional PDEs. This fact shows that all of the known integrable models can be considered as different deformations of some ODEs with Painlevé property. For instance, some (1 + 1)- and (2 + 1)-dimensional integrable sine-Gordon equations and Mikhailov-Dodd-Bullough equations can be considered as the deformations of a simple Riccati equation [19]. So we believe that higher dimensional, say, (3 + 1)-dimensional integrable models (if they exist), can also be obtained from lower dimensional ones.

In this Letter we try to propose a possible method to get higher dimensional integrable models from lower ones by means of the Painlevé analysis developed by Weiss, Tabor, and Carnevale (WTC) [20] in an extended form. Usually, the Painlevé analysis is used to study the singularity property, Bäcklund transformation (BT), symmetries, bilinearization, and other integrable properties and some types of exact solutions [20–22]. However, to my knowledge, no one uses the Painlevé analysis method to get new integrable models.

*General idea and assumption.*—For a given $n$-dimensional $N$ order PDE,

$$F(x_1, x_2, \ldots, x_n, t, u, u_{x_i}, u_{x_i x_j}, \ldots, u_{x_{i_1} x_{i_2} \ldots x_{i_N}}) = 0, \quad (1)$$

the model is called possessing the Painlevé property (PP) if all of the movable singularities of its solution with respect to an arbitrary singular manifold $\phi \equiv \phi(x_1, x_2, \ldots, x_n, t) = 0$ are poles. That is to say, by expanding the solution of (1) near the singular manifold

  

$\phi$, we should have the form

$$u = \sum_{j=0}^{\infty} u_j \phi^{j+\alpha}, \tag{2}$$

with $N - 1$ further arbitrary functions $u_j$ and negative integer $\alpha$. Substituting (2) into (1), all of the expansion coefficients $u_j$ can be solved as

$$u_j = f_j(x_i, \phi_{x_i}, \phi_{x_{i_1}x_{i_2}}, \ldots, u_{j_1}, u_{j_2}, \ldots, u_{j_{N-1}}) \quad (j \neq j_1, \ldots, j_{N-1}), \tag{3}$$

where $u_{j_1}, \ldots, u_{j_{N-1}}$ are arbitrary functions. If we take

$$u_j = 0, \qquad (j > -\alpha), \tag{4}$$

then we get a BT of (1), i.e., if $u_{-\alpha}$ is a solution of (1), then the truncated expansion $u$ given by (2) with (4) is also a solution of (1). Furthermore, if $u = 0$ is a solution of (1), the single soliton solution can be obtained from the BT and $u_{-\alpha} = 0$. Usually, when one uses the truncated expansions to discuss the Bäcklund transformation or exact solutions, no one discusses any properties of the equations $u_j = 0$, $(j \geq -\alpha)$, i.e.,

$$f_j \equiv f_j(x_i, \phi_{x_i}, \phi_{x_{i_1}x_{i_2}}, \ldots, u_{j_1} = 0, u_{j_2} = 0, \ldots, u_{j_{N-1}} = 0) = 0 \ (j \geq -\alpha). \tag{5}$$

Because equations of (5) for different $j$ must be consistent when we use the truncation expansion to study the integrable property of (1), we believe that many equations of (5) or some combinations of $u_j (j \geq -\alpha)$ may be integrable.

The integrable models obtained from (5) possess the same dimensions as the original model (1). In order to get some higher dimensional integrable models from (1), we may take the following two steps. First, we should embed the lower dimensional integrable model (1) in higher dimensions. In other words, we should consider that $u$ is not dependent only on the explicit independent variables $\{x_1, \ldots, x_n, t\}$ but also on some implicit ones, say, $\{x_{n+1}, \ldots, x_{n+m}\}$. Second, the Painlevé expansion (2) should be extended to a different resummation form such that the implicit independent variables $\{x_{n+1}, \ldots, x_{n+m}\}$ appear explicitly in the new expansion coefficients and the same number of arbitrary functions is still included in the new expansion form. The first step is quite trivial because infinitely many integral constants (inner parameters) can be included in the solutions of an arbitrary given PDE, and we can take these parameters as new independent variables. The second step can be realized because of the singular manifold $\phi$ being arbitrary. For instance, we can take

$$\xi \equiv \left( \frac{\phi_{x_{n+1}}}{\phi} - \frac{\phi_{x_{n+1}x_{n+1}}}{2\phi_{x_{n+1}}} \right)^{-1} \tag{6}$$

as a new expansion variable, i.e., we change the expansion (2) as

$$u = \sum_{j=0}^{\infty} u'_j \xi^{j+\alpha}, \tag{7}$$

with arbitrary $\xi$, $u'_{j_1}$, $u'_{j_2}, \ldots, u'_{j_{N-1}}$ and the same integers $j_1, j_2, \ldots, j_{N-1}$ and $\alpha$ as in (2). From Eq. (6) one can easily prove the following identities:

$$\xi_{x_i} = P_i - P_{ix_{n+1}}\xi + \tfrac{1}{2}(P_i S + P_{ix_{n+1}x_{n+1}})\xi^2, \quad i = 0, 1, 2, \ldots n, \ x_0 \equiv t, \tag{8}$$

where the functions

$$P_i \equiv \frac{\phi_{x_i}}{\phi_{x_{n+1}}}, \quad S \equiv \frac{\phi_{x_{n+1}x_{n+1}x_{n+1}}}{\phi_{x_{n+1}}} - \frac{3}{2}\left(\frac{\phi_{x_{n+1}x_{n+1}}}{\phi_{x_{n+1}}}\right)^2 \equiv \{\phi;\ x_{n+1}\} \tag{9}$$

are all invariant under the Möbious transformation

$$\phi \to \frac{a + b\phi}{c + d\phi}, \qquad ad \neq bc. \tag{10}$$

It is straightforward to see that all of the expansion coefficients $u'_j$ in (7) are all conformal invariant because they are functions of $P_i$, $S$, and $\{u'_{j_1}, u'_{j_2}, \ldots, u'_{j_{N-1}}\}$. Now a further independent variable $x_{n+1}$ has been included explicitly with the expansion coefficients $u'_j$ although the original equation (1) is not $x_{n+1}$ dependent explicitly. On the other hand, the conformal invariant plays very important roles in integrable theory. For instance, starting from the conformal invariance of the Korteweg–de Vries (KDV) and Kadomtsev-Petviashvili (KP) equations, one can obtain infinitely many symmetries, Darboux transformations, sine-Gordon extensions, etc. [23]. Starting from a quite general conformal invariant form, one can get infinitely many integrable models with the PP [24,25]. Now it is reasonable to assume that *if Eq. (1) is integrable, then the equations obtained from vanishing the coefficients of the Painlevé expansion (7), for $u'_j = 0$, $j \geq -\alpha$, are integrable.*

Although we have not yet proved this assumption and/or the assumption is not completely true, we can still obtain various higher dimensional integrable models from lower ones by checking the assumption for some given lower dimensional integrable models and small $j$. It is interesting that if the seed equation (1) is (2 + 1) dimensional, then the integrable models obtained from the assumption are (3 + 1) dimensional. Furthermore, the assumption can be used many times to get some integrable models in arbitrary dimensions no matter what the dimension of the seed model is. More concretely, we realize how to get some (3 + 1)-dimensional models from the KP, nonlinear Schrödinger (NLS), and Schwarz Korteweg–de Vries (SKDV) equations.

*(3 + 1)-dimensional integrable models from the KP, NLS, and SKDV equations.*—The KP equation ($x_0 = t$, $x_1 = x$, $x_2 = y$, $x_3 = z$)

$$(u_t - 6uu_x + u_{xxx})_x + 3\sigma^2 u_{yy} = 0 \tag{11}$$

is one of the most important integrable physical models widely used in many physical fields and studied by various mathematicians and physicists. The PP of the KP equation is first proven by WTC [20]. In order to obtain some nontrivial (3 + 1)-dimensional model from the KP equation, we restudy the PP of the model by using the new expansion (7) and consider that $u$ of (11) is not only a function of $\{x, y, t\}$ but also a function of $z$. As in the usual case [20], by means of the leading order analysis, we have

$$\alpha = -2, \qquad u'_0 = 2P_1^2. \tag{12}$$

Substituting (7) with (8) and (12) into (11) we get the recursion relation of the coefficients $u'_j$,

$$(j+1)(j-4)(j-5)(j-6)u'_j = f'_j(S, P_i, P_{ix_i}, \dots, u'_0, \dots, u'_{j-1}) \equiv f'_j, \tag{13}$$

where $f'_j$ is a complicated function of $u'_0, \dots, u'_{j-1}$ and the conformal invariants $P_i$, $S$, and their derivatives. After solving (13) one by one, $u'_j$ becomes only a function of the conformal invariants. The first two $u'_j$'s read

$$u'_1 = -2P_{1x} - 2P_1P_{1z}, \tag{14}$$

$$u'_2 = \frac{1}{6P_1^2}(3\sigma^2P_2^2 + P_1(P_0 + 4P_{1xx} + 2P_{1x}P_{1z}) + P_1^2(4P_{1xz} + P_{1z}^2) - 3P_{1x}^2 + 4P_1^4S + 4P_1^3P_{1zz}), \tag{15}$$

Using the relations (9), $u'_0, u'_1, u'_2, u'_3$ and the computer algebra, say, Maple or Mathematica, it is easy to prove that the three resonant conditions (13) for $j = 4, 5, 6$ are satisfied identically. The PP of the KP equation is reobtained under the new configuration.

Now using the assumption, we may obtain a set of integrable models

$$f'_j = 0, \qquad j = 2, 3, 7, 8, \dots, \tag{16}$$

where $j = 4, 5, 6$ disappear due to the resonance conditions at these values that are satisfied identically. It is difficult to prove the integrable property for the whole set. We can only check the integrabilities for small $j$ of (16). After substituting (9) into (16) for $j = 3$, one can see that the integrability for $u'_3 = 0$ is trivial because it is really the original (2 + 1)-dimensional KP equation in its Schwarz form. Although we can prove the PP of (16) for $j = 7$, we do not write it down because (16) with $j = 7$ is too complicated after substituting (9) in (about two or more printed pages). So we discuss only $j = 2$ here.

Substituting $j = 2$ and (9) into (16) yields a nontrivial (3 + 1)-dimensional model

$$\phi_z^4(4\phi_x\phi_{xxx} - 3\phi_{xx}^2 + \phi_x\phi_t + 3\sigma^2\phi_y^2) + 3\phi_x^4\phi_{zz}^2 - 6\phi_x^2\phi_z^2\phi_{xx}\phi_{zz} = 0. \tag{17}$$

In order to prove the integrability of (17), we change (17) into a variant form at first and then prove its PP in our new configuration. Equation (17) is in a hexalinear form. To prove the PP of a multilinear equation is much more difficult than to prove the PP in a nonhomogeneous form. By taking the exponential transformation

$$\phi = e^f, \tag{18}$$

we get a nonhomogeneous form of (17),

$$f_t + 4f_{xxx} - 2f_x^3 - 3f_{xx}^2f_x^{-1} + 3\sigma^2f_y^2f_x^{-1} + 3f_x^3f_{zz}^2f_z^{-4} - 6f_xf_{xx}f_{zz}f_z^{-2} = 0. \tag{19}$$

When $z = x = y$ or $f_y = f_{zz} = 0$, Eq. (19) reduces to an equivalent potential form of the modified KDV equation.

Multiplying (19) by $f_z^4f_x$ and using the leading order analysis, we find that there will be no algebraic poles [$\alpha = 0$ in (2)] in the Painlevé expansion. In order to include the algebraic poles in the Painlevé expansion,

$$\{f_x = U,\ f_y = V,\ f_z = W,\ f_t = G\} \tag{20}$$

is the simplest suitable transformation. Using Eq. (20) in (19) we get an equivalent equation system of (17),

$$4UW^4U_{xx} - 3W^4U_x^2 - 2W^4U^4 + 3U^4W_z^2 + UGW^4 - 6U^2W^2U_xW_z + 3\sigma^2V^2W^4 = 0, \tag{21}$$

$$U_t = G_x, \qquad V_t = G_y, \qquad W_t = G_z, \tag{22}$$

where Eqs. (22) are the consistent conditions of the transformation (20). To prove the PP of the equation system (21) and (22), we may use the traditional WTC approach [20] or its extended form proposed previously. Using the extended WTC approach we can get some more integrable models both in the (4 + 1) dimensions and in the (3 + 1) dimensions at the same time. So we take the extended approach here again. Expanding $U$, $V$, $W$, and $G$ as

$$U = \sum_{j=0}^{\infty} U_j\xi^{j+\alpha_1}, \qquad V = \sum_{j=0}^{\infty} V_j\xi^{j+\alpha_2},$$
$$W = \sum_{j=0}^{\infty} W_j\xi^{j+\alpha_3}, \qquad G = \sum_{j=0}^{\infty} G_j\xi^{j+\alpha_4}, \tag{23}$$

where $\xi$ is given by (6) with $n = 3$, $x_0 = t$, $x_1 = x$, $x_2 = y$, $x_3 = z$, and $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = -1$,

$$U_0 = \pm P_1, \qquad V_0 = \pm P_2,$$
$$W_0 = \pm P_3, \qquad G_0 = \pm P_0, \tag{24}$$

which can be determined by means of the leading order analysis and the relation (8) for $n = 3$.

Substituting (23) with $\alpha_i = -1$, (24) and (8) into (21) and (22), one can see that all of the needed five resonances are located at $j = -1, 1, 1, 1, 1$. The resonance at $j = -1$ is related to $\xi$ being arbitrary. The resonance conditions at $j = 1$ read

$$3U_0(P_1W_0 - U_0P_3)[W_0^2P_{1x_4} - U_0(W_0P_{3x_4} + W_{0z})] + 2W_0^3U_0P_{1x} + W_0^2(W_0P_1 - 3U_0P_3)U_{0x} - 2W_0[(2U_0^2W_0^2 - 3U_0^2P_3^2 + 3U_0W_0P_1P_3 - 2W_0^2P_1^2)U_1 - U_0(5P_1^2W_0 - 2W_0U_0^2 - 3P_1P_3U_0)W_1] = 0\,, \quad (25)$$

$$U_{0t} - G_{0x} + U_0P_{0x_4} - G_0P_{1x_4} = 0\,, \quad (26)$$

$$V_{0t} - G_{0y} + V_0P_{0x_4} - G_0P_{2x_4} = 0\,, \quad (27)$$

$$W_{0t} - G_{0z} + W_0P_{0x_4} - G_0P_{3x_4} = 0\,. \quad (28)$$

It is straightforward to see that all of the resonance conditions (25)–(28) are satisfied identically because of the relations (24) and (9) with $n = 3$. That is to say, five arbitrary coefficients $\xi$, $U_1$, $V_1$, $W_1$, and $G_1$ have been included in the expansion (23). Because Eq. (21) is a second order PDE and three equations in (22) are first order PDEs, the required number of arbitrary coefficients in the Painlevé expansion (23) is just five. So the equation systems (21) and (22) possess the PP and then Eq. (17) is a (3 + 1)-dimensional integrable model, meaning that it possesses conformal invariance and can be changed to a form with the PP.

Applying the assumption further to the systems (21) and (22), some (4 + 1)-dimensional models with the PP can be obtained. For instance, from $U_2 = V_2 = W_2 = G_2 = 0$ we get

$$(3\sigma^2P_2^2 + P_0P_1 - 2P_1^4S + P_1^2P_{1x_4}^2 + 2P_1P_{1x}P_{1x_4} - 3P_{1x}^2)P_3^4 - 6P_1^2P_3^3P_{3x_4}(P_1P_{1x_4} + P_{1x}) + 3(P_3^2P_{3x_4}^2 + P_{3z}^2 + 2P_3P_{3z}P_{1x_4})P_1^4 - 6P_1^2P_3^2P_{1x}P_{3z} - 6P_3^2P_1^3P_{3x}P_{1x_4} = 0\,, \quad (29)$$

where $P_i$ and $S$ are given by (9). The PP of (29) can be proved in the same way. Using the classical and nonclassical Lie approach, some kinds of (3 + 1)-dimensional integrable reductions can be obtained from (29).

The same idea can be used to get many higher dimensional integrable systems from other lower dimensional ones. For instance, the Painlevé integrable models

$$\phi_x[\phi_z(\phi_{zzt} + 2\phi_{zzz}) - 2\phi_{zz}\phi_{zt} - 3\phi_{zz}^2] + 2\phi_{xy}\phi_z^2 - \phi_t\phi_{xzz}\phi_z + 2\phi_{xz}(\phi_t\phi_{zz} - \phi_y\phi_z) = 0 \quad (30)$$

and

$$\frac{\phi_t}{\phi_x} + \{\phi;\, x\} + \frac{9}{2}\frac{\phi_x\phi_{zz}\phi_{xz}}{\phi_z^3} - \frac{3}{4}\frac{\phi_x^2\phi_{yy}^2}{\phi_y^4} + \frac{3}{8}\frac{\phi_x^2\phi_{zz}^2}{\phi_z^4} + 3\frac{\phi_x^2\phi_{yzz}}{\phi_z^2\phi_y} - \frac{3}{2}\frac{\phi_x\phi_{xyy}}{\phi_y^2} + 3\frac{\phi_x\phi_{xy}\phi_{yy}}{\phi_y^3} - \frac{9}{4}\frac{\phi_x\phi_{xzz}}{\phi_z^2} - 6\frac{\phi_x^2\phi_{yz}\phi_{zz}}{\phi_z^3\phi_y} = 0 \quad (31)$$

are obtained from the NLS equation $\{iu_t + u_{xx} - 4u^2v = 0,\ -iv_t + v_{xx} - 4v^2u = 0\}$ and the SKDV equation $\phi_t/\phi_x + \{\phi; x\} = 0$, respectively. Because of the simplicity and similarity, we omit the details on the derivation of (30) and (31) and the proof of their PP.

*Summary and discussion.*—In this Letter, after embedding the lower dimensional integrable models in higher dimensions and extending the standard WTC Painlevé expansion to a more general form such that the new expansion coefficients are conformal invariant, we have proposed an assumption to get more integrable models in higher dimensions from a known lower dimensional one. Starting from the KP, NLS, and SKDV equations, some (4 + 1)-dimensional and (3 + 1)-dimensional models with the PP and the conformal invariance are given explicitly.

Although the assumption is based on the fact that the original model is integrable and the resulting equations come from vanishing the consistent Painlevé expansion coefficients with the conformal invariance, we still have to check the integrability of the resulting models one by one because we have not yet proved the assumption strictly. Although the assumption has not yet been proved strictly, we have obtained many higher dimensional Painlevé integrable models from the assumption and there is no negative example found.

To reduce a higher dimensional model to some lower ones, one can use the classical and nonclassical Lie approaches or the direct method [15–17]. Now we know that it is also possible to get some higher dimensional models from lower ones.

The Painlevé integrable models obtained here should be studied further because real physical space is (3 + 1) dimensional and there is no knowledge of the (3 + 1)-dimensional soliton solutions.

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