Origin of Strong Scarring of Wave Functions in Quantum Wells in a Tilted Magnetic Field

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The anomalously strong scarring of the electronic wave functions of quantum wells in a tilted magnetic field is shown to be due to special properties of the classical dynamics of this system. A certain subset of periodic orbits is identified which is nearly stable over a very large interval of variation of the system parameters; hence this subset exhibits strong scarring. Semiclassical arguments shed further light on why these orbits dominate the experimentally observed tunneling spectra. [S0031-9007(97)04736-4]

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The localization of certain quantum wave functions in real space along unstable classical periodic orbits illustrates how quantum mechanics can violate the ergodic behavior expected from classical mechanics. Such wave functions are conventionally termed "scars" [1] and their properties have been extensively studied by theorists of quantum chaos during the past decade [2–4]. Recently, an experimental system has been discovered and studied [5,6] in which such scarred wave functions control to a large extent an observable physical property, the tunneling current through a double-barrier GaAs-AlGaAs heterostructure ("quantum well") under high bias. When a magnetic field is applied at an angle θ with respect to the normal to the barriers (the electric field direction), the resulting dynamics makes a transition to chaos [5–7] as θ is increased from zero. Calculations by Fromhold *et al.* [8] on the system found strong periodic scarring of quantum states, *and* that these scarred wave functions carried most of the tunnel current when the system was resonant. In the initial experiments [5,6] the level broadening (due to optic phonon emission) was too large to observe the resonances due to individual levels. However, in a later experiment [9] this was done, albeit at such low quantum numbers that the concept of scarring becomes less meaningful [10]. From extensive numerical [8–10] work we know (1) quantum states scarred by the same periodic orbit persist as the energy or magnetic field is varied so as to substantially increase the chaotic component in phase space. (2) The scars arise from only a few of the many unstable short periodic orbits in the tilted well. (3) These scars carry much of the resonant tunneling current for $\theta > 20°$.

The persistence of scarring by a single orbit as the degree of chaos in the classical dynamics is strongly varied is puzzling and untypical of chaotic systems. It is well known that scarred states are most likely to be associated with orbits which are not too unstable. A sufficient condition for strong scarring is that $\lambda T \lesssim 1$, where λ is the instability (Lyapunov) exponent associated with the orbit [1], and T is its period. In Hamiltonian systems unstable periodic orbits appear as marginally stable orbits at bifurcations; typically in chaotic systems (e.g., billiards) such orbits become monotonically more unstable as the classical parameter (such as energy) driving the system to chaos is increased. Therefore, such orbits only scar wave functions in the small interval of classical parameter space in which they are close to stability. Persistent scarring by the same orbit, as observed numerically in the tilted well, might then indicate that the association between small instability and scarring is violated in this system [11]. In fact, we will show in this Letter that a certain subset of unstable periodic orbits in the tilted well has an interesting classical "metastability," which allows them to satisfy the criterion $\lambda T \lesssim 1$ over a wide variation of the classical dynamics, thus resolving the puzzle. All of the orbits previously observed to cause strong scarring belong to this subset, which also contains new orbits whose strong scarring is demonstrated below. The theory also sheds further light on why these scarred states dominate the tunneling current.

We model the system by two infinite potential barriers corresponding to the x-y planes at $z = 0$ (the emitter) and $z = d$ (the collector), with an electric field $\mathbf{E} = -E\hat{z}$, and a "tilted" magnetic field $\mathbf{B} = B\cos\theta\hat{\mathbf{z}} + B\sin\theta\hat{\mathbf{y}}$ for $0 < z < d$ (see inset, Fig. 1). The classical Hamiltonian can be rescaled [12] so that the dynamics depends only on three dimensionless parameters: $\theta, \beta = 2Bv_0/E$, and $\gamma = \varepsilon/eV$, where V is the voltage across the well and $v_0 = (2\varepsilon/m^*)^{1/2}$ is the velocity corresponding to the total injection energy. In the experiments $\gamma \approx 1.17$ is constant to a good approximation [12] and so (at fixed tilt angle) β, the scaled magnetic field, is the single relevant variable. In Fig. 1 we show a typical surface of section taken at the collector barrier for the system at $\theta = 38°$, $\gamma = 1.17$, $\beta = 3.7$ which would correspond, e.g., to $B \simeq 4$ T, $eV \equiv eEd = 0.67$ V. Several stable and unstable period-three (three collisions with the collector barrier before repeating) orbits are present in a chaotic sea. A full theory of the bifurcation and stability properties of all the short periodic orbits (PO's) in this system can be found elsewhere [12] and we sketch only the most salient features here. Only PO's which collide with both the emitter and collector barriers contribute to the tunneling current, and we focus on these "emitter" orbits. Moreover,

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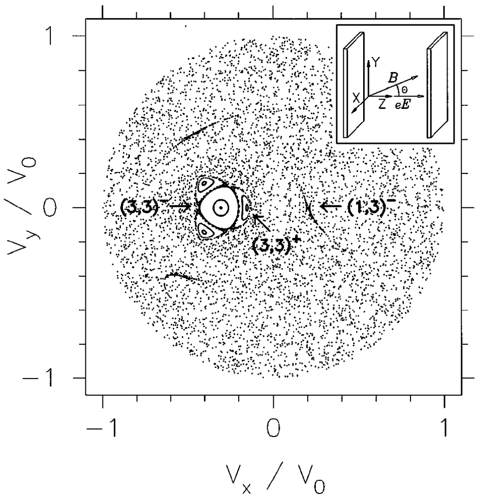

FIG. 1. Poincaré surface of section at the collector barrier for $\theta = 38°$, $\beta = 3.7$, $\gamma = 1.17$. The axes are the scaled velocities in the plane transverse to the electric field (see inset). The positions of one of the three fixed points of different period-three orbits are indicated by arrows (others are symmetrically placed). Note that a large period-one stable island near the center of the SOS also leads to strong localization of the quantum wave functions by the familiar procedure of local torus quantization. Because of its isolation from the relevant unstable PO's these stable contributions can be easily distinguished from scars. The inset shows a schematic of the geometry with our axis conventions.

optic phonon emission produces a temporal cutoff which allows only short periodic orbits to produce structure in the tunneling spectra.

Emitter orbits have the following properties [12]. Period-n orbits exist which collide with the emitter m times, where $m \leq n$. $m > n$ is forbidden since the emitter is at higher potential energy than the collector. Hence, we denote emitter orbits as (m, n), where $m \neq 0$. A crucial property of the emitter orbits in the tilted well is that they exist only for a finite interval of variation of the classical dynamics, appearing above a threshold value of $\beta = \beta_{c1}$ and disappearing at a higher value of $\beta = \beta_{c2}$ by an inverse bifurcation [usually a tangent bifurcation or a period-doubling bifurcation (PDB)]. A tangent bifurcation is one of the standard bifurcations of Hamiltonian systems in which two PO's which evolve due to the variation of a system parameter become degenerate and hence can disappear without violating conservation laws such as the Poincaré index theorem [13]. In a period-doubling bifurcation an existing orbit gives birth to a new orbit with the period equal to twice the period of the original orbit. Exactly at a PDB the new orbit is degenerate with the second repetition of the original orbit. At such degeneracy points the orbits must be marginally stable, which is expressed by the condition $|\mathrm{Tr}M| = 2$, where M is the stability matrix associated with the orbits. In the tilted well *all* emitter PO's disappear by such bifurcations.

While the "death" of the orbits follows the generic rules of Hamiltonian bifurcation theory [13], all the relevant orbits are "born" in a new kind of bifurcation, which we refer to as a cusp bifurcation (CB) [12,14]. CB's have nongeneric properties since they appear on a closed curve in the surface of section (SOS), where the Poincaré map describing the dynamics is nonanalytic. This "critical" curve separates initial conditions at the collector barrier which will reach the emitter on the next try from those which will not. Hence orbits originating just within the curve will receive a "kick" at the emitter, while those just outside will not. This leads to a discontinuity in the stability matrix M of any periodic orbit corresponding to a fixed point which crosses the boundary. In particular, all CB's occur by the simultaneous appearance of two new orbits, which infinitesimally above threshold differ by one point of contact with the emitter (this may correspond to either one or two fewer collisions). A similar discontinuity occurs near trajectories tangent to the circular reflector in the Sinai billiard [15] and leads to strong diffraction effects in the wave solutions.

We have shown [12] that of the two orbits born in a cusp bifurcation the orbit which reaches the emitter more times has diverging stability ($|\mathrm{Tr}M| \to \infty$) at β_{c1} while the other orbit in the pair can be either stable ($|\mathrm{Tr}M| < 2$) or unstable ($|\mathrm{Tr}M| > 2$) at β_{c1}. The latter case, in which two *unstable* orbits appear simultaneously, is forbidden for generic Hamiltonian systems [13]. The appearance of the relevant orbits away from marginal stability and in violation of the standard rules of bifurcation theory has made their appearance and subsequent evolution difficult to detect by numerical search.

The above results imply that only one partner in a cusp bifurcation is a candidate for strong scarring, the orbit that is born with fewer collisions with the emitter barrier. More detailed continuity arguments and numerical results [12] indicate that even though such orbits need not be marginally stable, near the cusp bifurcation they will have $|\mathrm{Tr}M| \approx 2$. Since these orbits are born near marginal stability and must die at marginal stability they have a special metastability which prevents them from becoming highly unstable as the chaos in the system increases. Hence we argue that these, and only these orbits, will scar repeatedly, as the chaos parameter, β, is varied.

Bifurcation and stability diagrams illustrating the behavior just described are shown in Fig. 2 for the case of period-two and period-three orbits. Among the period-two orbits the orbit denoted $(1,2)^-$ fits our criteria [the $-, +$ denoting (in-)stability; see caption]. At $\theta = 28°$ it is born in a CB with the higher connectivity orbit $(2,2)^-$ at $\beta \simeq 4.0$ and dies in an inverse TB with the orbit $(1,2)^+$ at $\beta \simeq 7.0$. This period-two orbit and another topologically similar orbit (not shown) which appears at a slightly higher value of β account very well [12] for the peak-doubling regions observed near $\theta = 28°$ in the experiments of Muller *et al.* [6]. This orbit was found to scar many wave

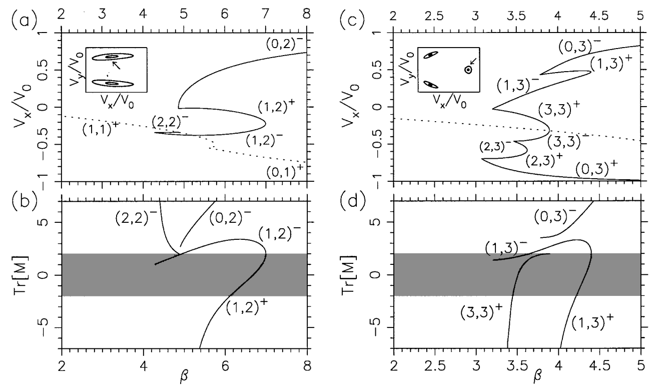

FIG. 2. Bifurcation diagrams for the relevant period-two (a) and period-three (c) orbits. The axis β (defined in text) may be regarded as the scaled magnetic field; the vertical axis is the v_x coordinate of one of the fixed points in the SOS associated with the orbit as indicated by the arrow in the inset. (b), (d) Plots of the trace of the monodromy (stability) matrix for the corresponding orbits; shaded area ($|\mathrm{Tr}[\mathrm{M}]| < 2$) denotes stability region. Orbits denoted $(1,2)^-, (1,3)^-$ remain slightly unstable over a large variation of β, leading to strong scarring. The notation $(m,n)^\pm$ used for the periodic orbits represents the topology of the orbit (m collisions with the emitter barrier and n collisions with the collector barrier per period; $+,-$ indicates that the orbit is stable, unstable just before its disappearance by inverse bifurcation).

functions in the work of Fromhold *et al.* [8]. Note from Fig. 2 that in roughly the same β interval there are two other period-two emitter orbits denoted $(2,2)^-, (1,2)^+$, each with rapidly varying stability. Both are born in CB's paired with an orbit with fewer collisions with the emitter; hence they are initially enormously unstable. Therefore they do not generate strong scars in the spectrum.

A similar story holds for one of the eight period-three orbits which appear around $\beta = 3.5$ at $\theta = 38°$. We denote this orbit $(1,3)^-$; its fixed points are indicated in Fig. 1 along with schematics of the orbit. This orbit has been discussed previously [16–18], particularly in connection with the observability of trifurcations, in the data of Ref. [6]. It is born in a CB as the partner of a $(3,3)^+$ orbit, remains near marginal stability for $3.2 < \beta < 4.4$, and dies in a TB with the $(1,3)^+$. Again, in the same interval there are several other unstable period-three emitter orbits which do not belong to the special metastable subset and hence do not scar strongly. Finally, by the same reasoning we have found a $(1,5)$ orbit which scars strongly.

Note that by our criteria the scarring orbits must always be (m,n) orbits with $m < n$; e.g., (1,2) can scar strongly, whereas (2,2) should not. On the other hand, it is easily shown [12] that as $\theta \to 0$ the *only* emitter orbits are of the type (n,n). Therefore, the interval of existence of the scarring orbits is small for small θ. Thus, for example, the period-three scarred states are unimportant for $\theta < 20°$.

We have tested this argument quantitatively by analyzing the quantum states of the tilted well for scars of the three orbits just identified [see Figs. 3(a)–3(c)]. By generating many spectra at different values of B, E we can search in the experimentally appropriate intervals of β with $1.1 < \gamma < 1.2$ and systematically detect these scars. In Figs. 3(d)–3(f) we plot a measure of scar strength versus action of the scarring orbit. As noted before [8,21], the energies ϵ_n of scarred states satisfy an approximate Bohr-Sommerfeld quantization rule, $S(\epsilon_n) = (n + \phi)2\pi\hbar$, so we expect and find a strong periodic modulation which turns on at the tangent bifurcation at which the orbit appears [19].

Finally, we comment on the fact that these scarred states tend to dominate the tunneling current at large tilt angles. All of the orbits studied here and elsewhere [8–10] which scar strongly have only a single collision with the emitter barrier. Since both emitter and collector surfaces of section must be symmetric when $v_y \to -v_y$, such orbits must have $v_y = 0$ at the emitter barrier [22]. Thus these orbits have unusually low transverse velocity at the emitter barrier compared to other orbits with the same periodicity in the collector Poincaré map. Since the emitter wave function is primarily a superposition of the first few Landau levels, the source of tunneling current has low transverse velocity and couples very well to these scarred states. The strong coupling of these particular scars to the

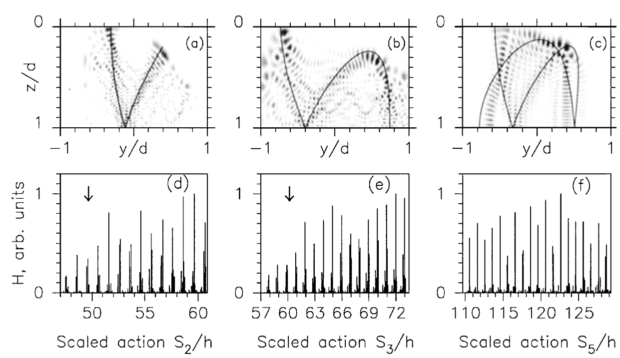

FIG. 3. (a)–(c) Examples of wave functions scarred by unstable $(1,2)^-$ orbit (a), $(1,3)^-$ orbit (b), and $(1,5)^-$ orbit (c); y-z projections of orbits are superimposed. (d)–(f) "Scar strength" H vs the scaled action $S(\epsilon_n)/h$ of the unstable orbit which scars the eigenstate of energy ϵ_n. The three cases shown are (a) $(1,2)^-$ orbit, (b) $(1,3)^-$ orbit, (c) $(1,5)^-$ orbit. The arrows indicate the values of β for the tangent bifurcation, which give birth to the periodic orbits. Note that when increasing action (or energy) the tangent bifurcation at higher $\beta \sim 1/\sqrt{\varepsilon}$ occurs at the *lower* action side. The peaks of the scar strength below the tangent bifurcation are due to the "ghost effect" [19,20]. Scaled actions below the bifurcation points were obtained by linear extrapolation of the (approximately linear) function $S(\varepsilon)/h$. Scar strength H is defined as the value of the Husimi function $H(y^0, p_y^0) \equiv \int dy dp_y\, W_n(y, p_y) \exp[-(y - y^0)/a_B^2 - a_B^2(p_y - p_y^0)^2/\hbar]$ [where $a_B = (\hbar/eB\cos\theta)^{1/2}$ is the effective magnetic length] calculated for the normal derivative of the wave function $[W_n(y, p_y) = \int d\Delta y \partial_z \Psi(y - \Delta y/2, 0) \partial_z \Psi(y + \Delta y/2, 0) \exp(i p_y \Delta y/\hbar)]$ at the location of the fixed point $(y^0 \propto v_x, p_y^0 \propto v_y)$ of the unstable periodic orbit.

emitter has been understood previously [9], and leads to quasiselection rules even far from the perturbative regime. This makes tunneling spectroscopy particularly suited to detecting scars in the tilted well.

We believe that the periodic orbit scenario described here of metastable orbits created in cusp bifurcations will be generic for nonintegrable Hamiltonian systems comprising the sum of smooth and hard-wall potentials. Hence the anomalous scarring found here may occur in a wide class of quantum-chaotic systems.

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