Nonstandard Diffusion Properties of the Standard Map

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The diffusion properties of the Chirikov-Taylor standard map are shown to be nonuniversal in the framework of the wave-particle interaction, because this map corresponds to a spectrum of waves whose initial phases are all correlated. The occurrence of diffusion is shown to be a peculiarity of the standard map when the average is made over the particles' initial positions. The force correlation time is shown to decrease more slowly with the wave amplitudes for the standard map than for waves with noncorrelated phases. Lévy flights are shown to be one more peculiarity of the standard map. [S0031-9007(98)06258-9]

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The Chirikov-Taylor map [1–3] defined by the finite difference equations $q_{n+1} - q_n = I_n$, $I_{n+1} - I_n = K\sin(q_{n+1})$, has been so widely studied during the past two decades that it is often referred to as the *standard map*. It is one of the basic models used in chaos theory and also has direct physical applications [4]. It is, in particular, relevant to the study of the statistical properties of the dynamics of a charged particle in a broadband spectrum, because it can be derived from the standard Hamiltonian

$$H = v^2/2 + (K/4\pi^2) \sum_{m=-\infty}^{+\infty} \cos(q - mt), \qquad (1)$$

which describes the motion of a particle of mass 1 in an infinite set of electrostatic waves having the same amplitudes, same wave numbers, zero initial phases, and integer frequencies. q_n is then defined by $q_n = q(t = 2\pi n)$ and I_n by $I_n = 2\pi v(t = 2\pi n)$.

In order to check the universality of the standard map regarding diffusion, we consider in this Letter the Hamiltonian

$$H' = v^2/2 + (K/4\pi^2) \sum_{m=-M}^{+M} \cos(q - mt + \varphi_m), \quad (2)$$

which has the same form as (1) but which includes only a finite number of waves whose initial phases φ_m are chosen independently. Choosing waves with initial random phases corresponds to the choice usually made in plasma physics, when dealing with a wave spectrum resulting from the unstable growth of a random noise. It is, for example, the case in the beam-plasma instability [5]. Yet, in chaos theory, such a physical situation is often modeled using the standard map [6].

The velocity distribution functions obtained from (1) and (2) are compared. Actually, in the case of the dynamics defined by (2), one can define two different velocity distribution functions: $f_q(v,t)$, when the statistics is made with respect to the particles' initial positions, and $f_\varphi(v,t)$, when the statistics is made over the initial phases, i.e., over the electric field realizations. It will be shown in this Letter that dynamical quantities averaged over the particles' initial positions, $\langle\,\rangle_q$, exhibit some nonremovable statistical noise, which is, however, smoothed out when the average is performed over the initial phases, $\langle\,\rangle_\varphi$. As a consequence, $f_q(v,t)$ only converges towards a noisy Gaussian, and therefore does not obey a diffusion equation in the case of the dynamics defined by (2), while $f_\varphi(v,t)$ does. In the case of the standard map one can define only the velocity distribution function $f_q(v,t)$ which will be shown to obey a diffusion equation for most values of K allowing large scale transport. Hence, when the statistics is made with respect to the particles' initial positions, diffusion is an atypical property of the standard map. Moreover, the force correlation time will be shown to decrease more slowly with K in the case of the standard map than in the case of the dynamics defined by (2). Finally, it will be shown that, because the initial phases φ_m are not all correlated in H', the dynamics defined by (2) cannot exhibit any Lévy flights, in contrast to the case of the standard map.

We consider here only the case where the initial velocity distribution function, $f(v,0)$, is a Dirac distribution. Then, if there is diffusion, i.e., if $\partial f(v,t)/\partial t = D\partial^2 f(v,t)/\partial v^2$, $f(v,t)$ is a Gaussian of variance $2Dt$. In the case of the standard Hamiltonian, when $\langle \Delta v^2(t)\rangle_q$ is numerically observed to evolve linearly with time, the numerical distribution function $f_q(v,t)$ indeed converges towards a Gaussian, as expected. This can be seen by comparing $f_q(v,t)$ to a Gaussian of the same variance, like in Fig. 1(*a*), or by using a statistical test, like the Kolmogorov-Smirnov (KS) test [7], which yields the probability that a given distribution function would be a Gaussian. In the case of Fig. 1(*a*), the KS test indicates that, with a probability equal to 85%, $f_q(v,t)$ is a Gaussian.

Surprisingly enough, the convergence of $f_q(v,t)$ towards a Gaussian is actually an atypical result as such a convergence is not observed in the case of the Hamiltonian (2). This is illustrated in Fig. 1(*b*), and the KS test indicates a probability only equal to 30% that $f_q(v,t)$ would

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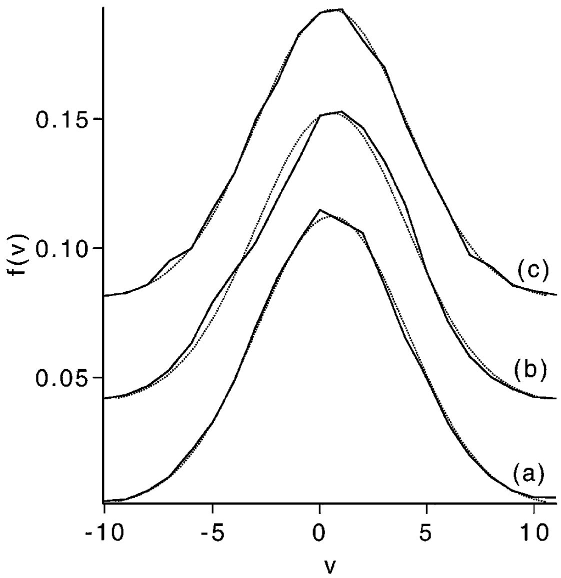

FIG. 1. Numerical distribution functions at time t (solid lines) compared to the Gaussian distributions of the same variances (dotted lines): (a) standard map, $K = 32$, $t = 90(K/4\pi^2)^{-2/3} \simeq 104.4$, 9000 particles' initial positions; (b) Hamiltonian (2), $K = 32$, $t = 900(K/4\pi^2)^{-2/3} \simeq 1044$, 90 000 particles' initial positions; (c) Hamiltonian (2), for the same K and t as in (a), and 9000 phase realizations.

be a Gaussian. Yet, as shown in [8], $\langle \Delta v^2(t) \rangle_q$ is indeed numerically observed to evolve linearly with time, even for times shorter than the one corresponding to Fig. 1(b). $f_q(v,t)$ actually looks like a noisy Gaussian even though the number of samples and the time of integration corresponding to Fig. 1(b) are 10 times higher than in Fig. 1(a). Actually, the amplitude of the noise in Fig. 1(b) is found to be the same in a velocity distribution function corresponding to a time twice shorter, or when the number of samples is divided by 10. Therefore, this noise does not decrease as time goes on or if the number of samples is increased: it is a nonremovable noise. This implies that, for the dynamics defined by (2), when the statistics is made with respect to the particles' initial positions, there is no diffusion.

However, a statistics made over the initial phases φ_m yields a diffusion. $f_\varphi(v,t)$ is indeed numerically observed to converge towards a Gaussian as can be seen in Fig. 1(c), for the same values of K and time, and the same number of samples as in Fig. 1(a). Moreover, the KS test also indicates a probability of 85% that $f_\varphi(v,t)$ would be a Gaussian. We thus conclude that, in the case of the Hamiltonian (2), *chaotic diffusion is an average-dependent statistical property.*

This can easily be understood using a property of locality, recently introduced in [9]. Using perturbation theory, it was shown in [9] that, when the φ_m are chosen independently, the statistical properties of the dynamics defined by (2) are the same as those of the reduced dynamics which encompasses only the waves with phase velocities m such that $|m - v(t)| \leq \Delta v$, where Δv is proportional to $K^{2/3}$. Therefore, the reduced dynamics is a dynamical system which changes as the particle's velocity changes. Actually, as the wave phases are independent variables, the reduced dynamics changes in an incoherent way. In particular, the reduced dynamics related to velocities v_1 and v_2 separated by $2\Delta v$ at least are completely uncorrelated. These two reduced dynamics thus induce completely independent changes in the particle's velocity, and also in the particle's position. As shown in [9], this last feature prevents any recorrelation when the particle's velocity assumes once again the value v_1 after having been equal to v_2. Therefore, the particle experiences a sum of uncorrelated increments of velocity. Then, because of the central limit theorem, the velocity distribution function obtained by averaging over the realizations of these uncorrelated increments of velocity converges towards a Gaussian. As the increments of velocity are uncorrelated because the φ_m are independent variables, averaging over the φ_m is equivalent to averaging over the realizations of the increments of velocity. This explains why the velocity distribution function $f_\varphi(v,t)$ converges towards a Gaussian [9]. Now, it can easily be seen from the Hamilton equations of (2) that, at any time t, the values of a particle's position and velocity are uniquely determined once the value of $q(0) + \varphi_m$ is prescribed for any m such that $-M \leq m \leq M$. This implies that the set of parameters defining the dynamics of (2) is of dimension $2M + 1$. In the space of these parameters, the point whose mth coordinate is $q(0) + \varphi_m$ moves along a straight line when only $q(0)$ is varied. Therefore, averaging over the initial position, $q(0)$, only amounts to visiting a straight line, and thus a set of dimension one, and of measure zero, in the set of all the parameters. This is not enough to obtain smooth quantities, and explains the statistical noise observed in Fig. 1(b).

For the sake of simplicity, we carried out all of the above discussion in the particular case where the φ_m are independent. However, the results shown previously remain valid in the case where the φ_m only depend on each other over a finite range, i.e., if there exists an integer l_c, $l_c \ll M$, such that φ_i is independent of any φ_j such that $|i - j| \geq l_c$. Therefore, as regards transport properties, the standard map is only relevant to the description of those wave spectra where the initial phases of almost all the waves are correlated.

In order to further exemplify the difference between the averages made with respect to the particles' initial positions and the averages over the phase realizations, let us investigate the evolution of $\langle \Delta v^2(t) \rangle$. A direct calculation [9] of $\langle \Delta v^2(t) \rangle_\varphi$, using the Hamilton equations of (2), shows that during a K-independent time t_0, shorter than 2π, $\langle \Delta v^2(t) \rangle_\varphi \simeq 2D_{\mathrm{QL}}t$, where $D_{\mathrm{QL}} = K^2/32\pi^3$ is the so-called quasilinear value of the diffusion coefficient [10] (see Fig. 2). Plotting the initial evolution of $\langle \Delta v^2(t) \rangle_q$ for the Hamiltonian (2) shows that this curve is far from being a straight line (see Fig. 2), and is actually a very noisy curve. As for $f_q(v,t)$, the noise present in $\langle \Delta v^2(t) \rangle_q$

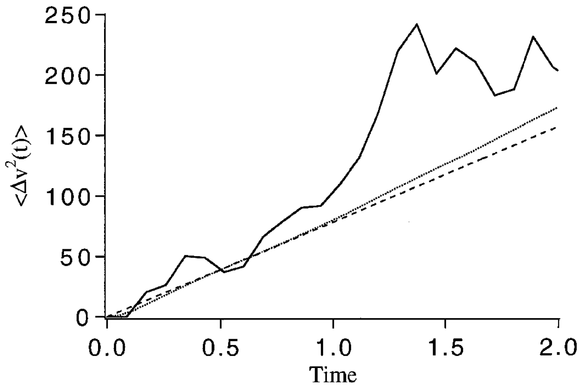

FIG. 2. $\langle \Delta v^2(t) \rangle_q$ (solid line) and $\langle \Delta v^2(t) \rangle_\varphi$ (dotted line) for the Hamiltonian (2) with $K = 200$. The dashed straight line has the quasilinear slope.

comes from the fact that averaging over the particles' initial positions amounts to averaging over a very small fraction of the parameters defining the dynamics of (2). The strong discrepancy between the initial evolutions of $\langle \Delta v^2(t) \rangle_q$ and $\langle \Delta v^2(t) \rangle_\varphi$, shown in Fig. 2, is a good illustration of how different the statistical properties of a dynamics can be, depending on the averaging process.

In the case of the standard map, investigating $\langle \Delta v^2(t) \rangle$ for times shorter than 2π does not make any sense because, if $t < 2\pi$, $\langle \Delta v^2(t) \rangle = 0$. However, studying $\langle \Delta v^2(t) \rangle$ for longer times enables one to estimate the force correlation time. Indeed, when the force is decorrelated, $\langle \Delta v^2(t) \rangle$ evolves linearly. Studying the long-time evolution of $\langle \Delta v^2(t) \rangle$, we now compare the force correlation times for the Hamiltonians (1) and (2).

In the case of the Hamiltonian (2), because of the property of locality, force decorrelation can actually be deduced from the study of the reduced force $F(t) = (K/4\pi^2) \sum_{|m-v(t)| \leq \Delta v} \cos(q - mt + \varphi_m)$. As already mentioned, after having moved by more than $2\Delta v$ along the velocity axis, a particle is acted upon by a reduced force $F(t)$ independent of any of the phases φ_m present in $F(0)$. This implies that $\langle F(t)F(0) \rangle_\varphi \simeq 0$. When the statistics is performed with respect to the phase realizations, the force is thus expected to decorrelate when the particle has moved along the velocity axis by an amount close to $2\Delta v$. In order to numerically test this result, one needs to clearly specify a definition of the force correlation time, because there is actually no time τ_c such that when $t \geq \tau_c$ the force correlation function is exactly 0. Therefore, $\langle \Delta v^2(t) \rangle_\varphi$ never evolves exactly linearly. Moreover, in a numerical simulation, the use of a finite number of samples to perform the averages entails a statistical noise in $\langle \Delta v^2(t) \rangle_\varphi$ which prevents its exact evaluation. In the simulations we performed, the relative amplitude of the fluctuations of $\langle \Delta v^2(t) \rangle_\varphi$ was less than 5%. This led us to define the force correlation time, τ_c, as the time such that, for $t \geq \tau_c$, the relative discrepancy between $\langle \Delta v^2(t) \rangle_\varphi$ and its best linear fit always remains below 5%. We then found that the force indeed decorrelates when a particle has moved by a quantity close to $2\Delta v$ along the velocity axis. In order to illustrate this result, we estimated the values C and D which best fit the linear evolution $\langle \Delta v^2(t) \rangle_\varphi = 2Dt + C$, and then plotted $\langle \Delta v^2(t) - C \rangle_\varphi / 2Dt$ as a function of $N_b(t) = \langle v_{\max}(t) - v_{\min}(t) \rangle / 2\Delta v$, where $\Delta v = 0.43 K^{2/3}$. One can see in Fig. 3 that $\langle \Delta v^2(t) - C \rangle_\varphi / 2Dt \simeq 1$ when $N_b \simeq 1$. Using the scaling properties analytically derived in [9], this implies that the force correlation time scales as $K^{-2/3}$.

If we now consider the case where the averages are performed with respect to the particles' initial positions then, when calculating $\langle F(t)F(0) \rangle_q$, where $F(t)$ is still the reduced force, the fact that the phases present in $F(t)$ and $F(0)$ are the same or not does not make any difference. Actually, $\langle F(t)F(0) \rangle_q \simeq 0$ only if $q(t)$ can be considered as independent from its initial value. Hence, force decorrelation is only due to the incoherent change of the position, $q(t)$, which begins to occur after a particle has moved by more than $2\Delta v$ along the velocity axis. Therefore, unlike in the case when the average is performed with respect to the phases φ_m, a shift by $2\Delta v$ along the velocity axis only has an indirect consequence on force decorrelation. The force is thus expected to decorrelate later when the statistics is made with respect to the particles' initial positions than when the averages are performed with respect to the phase realizations. This is confirmed numerically, as the force decorrelates when $N_b \simeq 2.5$ when the average is made over the particles' initial positions, instead of $N_b \simeq 1$ when the average is made over the phase realizations (see Fig. 3).

In the case of the standard map, the way the force correlation time, τ_c, scales with K cannot be directly estimated by studying the evolution of $\langle \Delta v^2(t) \rangle_q$. This is due to the fact that the force decorrelates after only a few map iterations, which entails a too large imprecision on τ_c to derive a scaling. However, one can have some indications about the way the force correlation time decreases with K by studying how fast the diffusion coefficient converges

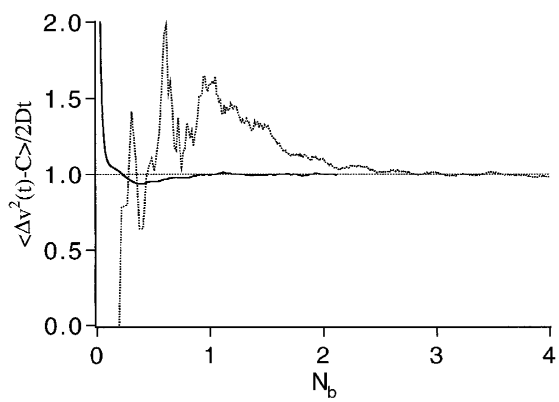

FIG. 3. $\langle \Delta v^2(t) - C \rangle / 2Dt$ for the Hamiltonian (2) with $K = 200$: average over the phases (solid line); average over the initial positions (dotted line).

towards its quasilinear value when $K \to \infty$. Indeed, using our definition of the force correlation time, when $t \geq \tau_c$, the slope of $\langle \Delta v^2(t) \rangle$, which yields an estimate of the diffusion coefficient, does not vary by more than 5%. For instance, when $\tau_c = 2\pi$, the diffusion coefficient for the standard map differs from its quasilinear value by less than 5%. When $K = 8$, we could estimate that the force gets decorrelated after a number of iterations n such that $5 \leq n \leq 7$. Therefore, $10\pi \leq \tau_c \leq 14\pi$. If τ_c scaled as $K^{-2/3}$, the value K_{QL} of the stochastic parameter corresponding to $\tau_c = 2\pi$ would be such that $89 \leq K_{\mathrm{QL}} \leq 148$. Now, it has been analytically estimated [3,11] that the relative discrepancy between the diffusion coefficient of the standard map and the quasilinear diffusion coefficient is $\sqrt{8/(\pi K)} \cos(K - 5\pi/4)$. This relative discrepancy is less than 5% for any $K \geq 3200/\pi \simeq 1020$. This value is much larger than the one estimated by assuming $\tau_c \sim K^{-2/3}$. Therefore, τ_c decreases with K more slowly than $K^{-2/3}$. Actually, if $\tau_c \sim K^{-\alpha}$, then we estimate $0.33 \leq \alpha \leq 0.4$. This result is thus different from the one corresponding to the Hamiltonian (2).

So far we have compared the diffusion properties of the Hamiltonians (1) and (2). Let us now investigate the universality of the nondiffusive properties of the standard map. It is well known that, regardless of how large K^* is, in the interval $[K^*, +\infty]$ there exist ranges of values of the stochasticity parameter K for which $\langle \Delta v^2(t) \rangle$ does not evolve linearly, even for long times. This is due to the existence of accelerator modes [1] which give rise to islets of stability in phase space. When a particle is started inside such an islet, it experiences an almost constant acceleration, and therefore not a diffusive process. Moreover, these islets of stability are very sticky, which implies that if a particle's orbit comes close to one of them, the particle experiences a coherent acceleration for a long time. This gives rise to Lévy flights [12], implying a time dependence of $\langle \Delta v^2(t) \rangle$ of the form $\langle \Delta v^2(t) \rangle \sim t^\gamma$, with $\gamma > 1$, instead of $\gamma = 1$ for diffusion. The occurrence of Lévy flights can be encountered in various systems (see references in [12]); however, in the context of the wave-particle interaction, it is a peculiarity of the standard map. Indeed, the previous section showed that, in the case of the Hamiltonian (2), the force decorrelates whatever the value of K allowing transport on a scale of the order of M.

In conclusion, we showed that the diffusion properties of the standard map do not generalize to a wave spectrum with uncorrelated phases. Unlike in the case of the standard map, for the dynamics defined by (2) $\langle \Delta v^2(t) \rangle$ cannot evolve in a superdiffusion way. When $\langle \Delta v^2(t) \rangle$ is numerically observed to evolve in a diffusionlike fashion, the force correlation time was shown to decrease more slowly for the standard map than for the Hamiltonian (2). Moreover, the convergence of the velocity distribution function towards a Gaussian was shown to be a peculiarity of the standard map when the statistics is made with respect to the particles' initial positions. When the wave initial phases are not all correlated, such a convergence only occurs when the statistics is made with respect to the field realizations. Chaotic diffusion is thus an average-dependent property. In particular, the evolution of $\langle \Delta v^2(t) \rangle$ was shown to strongly depend on the averaging procedure.

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