Wilson Loops in Large N Field Theories

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(Received 19 March 1998)

We propose a method to calculate the expectation values of an operator similar to the Wilson loop in the large N limit of field theories. We consider $\mathcal{N} = 4$ (3 + 1)-dimensional supersymmetric Yang-Mills theory. The prescription involves calculating the area of a fundamental string world sheet in certain supergravity backgrounds. We also consider the case of coincident M-theory five-branes where one is led to calculating the area of M-theory two-branes. We briefly discuss the computation for (2 + 1)-dimensional supersymmetric Yang-Mills theory with 16 supercharges which is nonconformal. In all of these cases, we calculate the energy of a quark-antiquark pair. [S0031-9007(98)06198-5]

PACS numbers: 11.25.Hf, 11.15.Pg

It has been expected for some time that the 't Hooft limit [1] of large N gauge theories is related to a string theory (see [2] and references therein). In [3], a precise string theory was proposed for the 't Hooft limit of $\mathcal{N} = 4$ supersymmetric Yang-Mills theory in (3 + 1) dimensions, based on earlier studies [4]. The 't Hooft limit is defined as the limit of $N \to \infty$ keeping $g_{\rm YM}^2 N$ fixed. In this limit, we get a weakly coupled string theory on $AdS_5 \times S^5$ where the radius of the five sphere and the curvature radius of anti–de Sitter are proportional to $(g_{\rm YM}^2 N)^{1/4}$ in string units. There is also a flux of the Ramond-Ramond self-dual five-form field strength on the five sphere. The string coupling is $g \sim g_{\rm YM}^2$ and goes to zero in the 't Hooft limit. In general, we do not know how to solve free string theory on $AdS_5 \times S^5$. However, when gN is large the radius of curvature is large, and we can use the string in background fields approximation. In [5–7], it was shown how to calculate conformal dimensions of operators and correlators in conformal field theory in terms of supergravity when gN is large. In this paper, we consider the problem of calculating the expectation values of Wilson loop operators. The proposal is that these expectation values correspond to the area of a world sheet whose boundary is the loop in question. We will further consider similar observables for the $M5$-brane theory [the conformal $(0,2)$ six-dimensional theory]. We also discuss Wilson loops in nonconformal theories associated with D-two-branes.

The Wilson loop.—Consider a Yang-Mills theory. The Wilson loop operator is

$$W(C) = \frac{1}{N} \operatorname{Tr} P e^{i \oint_C A}, \tag{1}$$

where C denotes a closed loop in spacetime, and the trace is over the fundamental representation. We will be considering mostly the Euclidean field theory. We can view the Wilson loop as the phase factor associated to the propagation of a very massive quark in the fundamental representation of the gauge group. A loop that is often considered is a rectangle with one direction along the time direction of length T and the other direction of length L. From the expectation value of this rectangular Wilson loop, it is possible to read off the energy of a quark-antiquark pair. Namely, in the limit $T \to \infty$ the expectation value of the Wilson loop is

$$\langle W(C) \rangle = A(L) e^{-TE(L)}, \tag{2}$$

where $E(L)$ is the energy of the quark-antiquark pair.

In order to perform this calculation for the cases of interest, it will be necessary to introduce massive quarks. To this effect consider breaking $U(N+1) \to U(N) \times U(1)$ by giving some expectation value $\vec{\Phi}$ to a Higgs field. Then the massive W bosons have a mass proportional to $|\vec{\Phi}|$ and transform in the fundamental representation of $U(N)$. So in the limit $|\vec{\Phi}| \to \infty$ they provide the very massive quarks necessary to compute Wilson loops in the $U(N)$ theory. Notice that we are interested in physics for energy scales much lower than $|\vec{\Phi}|$ so that the $U(N)$ theory is effectively decoupled from the $U(1)$ theory. Consider the equation of motion for the massive W boson. Extracting the leading time dependence as $W = e^{-i|\Phi|t}\tilde{W}$, we get an equation for $\tilde{W}$ which to first order in $1/|\vec{\Phi}|$ reads

$$(\partial_0 - iA_0 - i\theta^I X^I)\tilde{W} = 0, \tag{3}$$

where we have defined $\theta^I \equiv \Phi^I/|\vec{\Phi}|$. Notice that A_0 and X_I are matrices in the adjoint of $U(N)$. This implies that if we consider this massive W boson describing a closed loop C its interaction with the $U(N)$ gauge field will lead to the insertion of the operator

$$W(C) = \frac{1}{N} \operatorname{Tr} P e^{i \oint ds [A_\mu(\sigma)\dot{\sigma}^\mu + \theta^I(s) X^I(\sigma)\sqrt{\dot{\sigma}^2}]}. \tag{4}$$

The difference with (1) is the fact that we have an extra coupling to X^I. The operator in (4) is determined by the contour C [or $\sigma^\mu(s)$] as well as a function $\vec{\theta}(s)$ mapping each point on the loop to a point on the five sphere. We are interested in this operator because it is the one that naturally arises when we consider the propagation of a massive W boson. The appearance of X^I might seem surprising at first sight, but it is obvious when we remember that a string ending on a p-brane is not only a source of electric field but it also carries "scalar" charge for the fields X^I since it is pulling the brane. In fact, this coupling is crucial to understand the Bogomol'nyi-Prasad-Sommerfield (BPS) bound for strings stretching between different branes [8]. In the calculations below, $\theta(s)$ will be basically constant.

Relation to supergravity.—A natural proposal for the expectation value of the Wilson loop is

$$\langle W(C)\rangle \sim e^{-S}, \tag{5}$$

where, in the large gN approximation, S is the proper area of a fundamental string world sheet which at the boundary of AdS describes the loop C and lies along $\theta^I(s)$ on S^5 (see Fig. 1). In general, we should consider the full partition function of string theory on $AdS_5 \times S^5$ with the condition that a string world sheet is ending on the loop C and the points $\vec{\theta}(s)$ on S^5 at the boundary of AdS. This is a natural proposal in terms of the identification proposed in [5,7] for relating gauge theory observables to calculations on AdS. However, the right-hand side in (5) contains also the contribution from the mass of the W boson, and it is therefore infinity. Subtracting this contribution, we find a finite result for the Wilson loop operator

$$\langle W(C)\rangle \sim \lim_{\Phi\to\infty} e^{-(S_\Phi - \ell\Phi)}, \tag{6}$$

where ℓ is the total length of the Wilson loop, measured with the flat Minkowski metric appropriate to the gauge theory, and Φ is the mass of the W boson. Equation (6) is our final recipe for computing the Wilson loop. This result is not "zigzag" invariant, in the sense of [2], since the operator (4) is not invariant, as opposed to (1).

Quark-antiquark potential.—In this section, we consider the calculation of the rectangular Wilson relevant to extract the quark-antiquark potential. We take the angle $\theta^I(s) = \theta^I_0$ to be a constant. We consider the limit $T \to \infty$. In this limit, the problem becomes translational invariant along the $\hat{T}$ direction. We put the quark at $x = -L/2$ and the antiquark at $x = L/2$. Here "quark" means an infinitely massive W boson connecting the N branes with one brane which is far away in the direction $\vec{\theta}_0$. The action for the string world sheet is

$$S = \frac{1}{2\pi\alpha'}\int d\tau\, d\sigma \sqrt{\det G_{MN}\,\partial_\alpha X^M \partial_\beta X^N}\,, \tag{7}$$

where G_{MN} is the Euclidean $AdS_5 \times S^5$ metric

$$ds^2 = \alpha'\left[\frac{U^2}{R^2}(dt^2 + dx_i\, dx_i) + R^2\frac{dU^2}{U^2} + R^2\, d\Omega_5^2\right], \tag{8}$$

where $R = (4\pi gN)^{1/4}$ is the radius in string units, and $U = r/\alpha'$ has dimensions of energy. Notice that the factors of α' cancel out in (7), as they should. Since we are interested in a static configuration, we take $\tau = t$, $\sigma = x$ so that the action becomes

$$S = \frac{T}{2\pi}\int dx \sqrt{(\partial_x U)^2 + U^4/R^4}\,. \tag{9}$$

We need to solve the Euler-Lagrange equations for this action. Defining U_0 to be the minimum value of U, which by symmetry occurs at $x = 0$, we find that the solution is (all integrals below can be calculated in terms of elliptic of beta functions)

$$x = \frac{R^2}{U_0}\int_1^{U/U_0}\frac{dy}{y^2\sqrt{y^4 - 1}}\,, \tag{10}$$

where U_0 is determined by the condition

$$\frac{L}{2} = \frac{R^2}{U_0}\int_1^\infty \frac{dy}{y^2\sqrt{y^4 - 1}} = \frac{R^2}{U_0}\frac{\sqrt{2}\,\pi^{3/2}}{\Gamma(1/4)^2}\,. \tag{11}$$

The qualitative form of the solution is shown in Fig. 2. Notice that the string approaches the point $x = L/2$ quickly for large U, $L/2 - x \sim 1/U^3$.

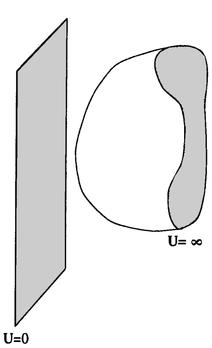

FIG. 1. Proposal to calculate Wilson loop expectation values. We should consider the partition function of string theory on $AdS_5 \times S^5$ with a string world sheet ending on the contour C on the boundary of AdS.

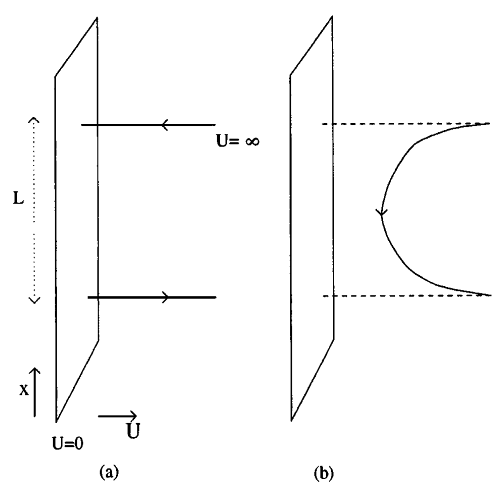

FIG. 2. (a) Initial configuration corresponding to two W bosons before we turn on their coupling to the U(N) gauge theory. (b) Configuration after we consider the coupling to the U(N) gauge theory. This configuration minimizes the action. The quark-antiquark energy is given by the difference of the total length of the strings in (a) and (b).

Now we compute the total energy of the configuration. If we just plug in the solution (10) in (9), we find that the answer is infinity. However, as we said above, this infinity is simply due to the fact that we are including the mass of the W boson which corresponds to a string stretching all the way to $U = \infty$. We can regularize the expression by integrating the energy only up to $U_{\max}$. Subtracting the regularized mass of the W boson which is $U_{\max}/2\pi$ we find a finite result

$$E = -\frac{4\pi^2(2g_{\rm YM}^2 N)^{1/2}}{\Gamma(1/4)^4 L}. \qquad (12)$$

We see that the energy goes as $1/L$, a fact which is determined by conformal invariance. Notice that the energy goes as $(gN)^{1/2}$ as opposed to gN which is the perturbative result. This indicates some screening of the charges. The above calculation makes sense for all distances L when gN is large independently of the value of g: This suggests that one could define a magnetic Wilson loop operator which for large gN would be determined in terms of classical D-string solutions with prescribed boundary conditions at infinity. In the standard 't Hooft limit, the interaction between Wilson loops is governed by g which goes as $1/N$.

Case of nonconstant angle.—Now we consider the case where the "angle" of the two quarks is different. This arises when we break $\mathrm{U}(N+2) \to \mathrm{U}(N) \times \mathrm{U}(1)_1 \times \mathrm{U}(1)_2$ by giving expectation values $\vec{\Phi}_1$, $\vec{\Phi}_2$ to the two U(1) factors. Then the angles are $\vec{\theta}_i = \vec{\Phi}_i/|\vec{\Phi}|$. So we consider a W boson described by a string going between the N branes and the brane associated to $\mathrm{U}(1)_1$ and a W boson going between the brane associated to $\mathrm{U}(1)_2$ and the N branes. Notice that the orientation of the string determines whether we have a quark or an antiquark. The potential for this configuration can be calculated in terms of the large T limit of the expectation value of the rectangular Wilson loop with different values of $\vec{\theta}$ on each timelike direction. So we should consider a string world sheet which at $x = L/2$ goes to $U = \infty$ and to the point $\vec{\theta}_1$ of the five sphere and at $x = -L/2$ goes to $U = \infty$ and to the point $\vec{\theta}_2$ of the five sphere. The action for a time independent configuration is

$$S = \frac{T}{2\pi}\int dx\sqrt{(\partial_x U)^2 + U^2(\partial_x\vec{\theta})^2 + U^4/R^4}. \qquad (13)$$

From the symmetries of the problem we see that the string will lie along a great circle of the sphere. So if we call θ the angle along this great circle we can choose $\theta_{1,2} = \pm\Delta\theta/2$. The problem then becomes symmetric around $x = 0$. We can solve the Euler-Lagrange equations as above by using the fact that the Lagrangian (13) is independent of x and θ so that we have conserved quantities associated with "energy" and "angular momentum" (interpreting x as time). Solving these equations, we find

$$x = \frac{R^2}{U_0}\sqrt{1-l^2}\int_1^{U/U_0}\frac{dy}{y^2\sqrt{(y^2-1)(y^2+1-l^2)}},$$
$$\theta = l\int_1^{U/U_0}\frac{dy}{\sqrt{(y^2-1)(y^2+1-l^2)}}, \qquad (14)$$

and the parameters U_0, l are determined by the conditions

$$\frac{L}{2} = x(U=\infty) = \frac{R^2}{U_0}\sqrt{1-l^2}\,I_1(l),$$
$$\frac{\Delta\theta}{2} = \theta(U=\infty) = l I_2(l), \qquad (15)$$

where $I_i(l)$ are defined to be the integrals in (14) with the upper limit being infinity. We can also calculate the energy of the system, subtracting the mass of the W bosons, and we find

$$E = -\frac{2}{\pi}\frac{(2g_{\rm YM}^2 N)^{1/2}}{L}(1-l^2)^{3/2}I_1^2(l), \qquad (16)$$

where l is a function of the angle (15). It is interesting to notice that when $\Delta\theta \to \pi$ then $l \to 1$. Then the solution looks like two straight strings going down to $U = 0$ and the energy (16) goes to zero, as expected since this is a BPS configuration.

M-theory membranes.—If we study the theory of coincident M-theory five-branes, the $(0,2)$ conformal field theory in six dimensions [9], we are led to consider M-theory on $AdS_7 \times S^4$. In this case, one could define Wilson "surface" observables [10]. Since we do not have an explicit formulation of the theory, we do not have a formula analogous to (4). However, we could define the Wilson "surfaces" as the phase factor associated with the propagation of a very heavy string on branes (subtracting the part proportional to the free propagation of the heavy string). In order to be more precise, let us suppose that we start with $N+1$ branes and then we Higgs by separating one of the branes. A membrane stretched between the N five-branes at the origin and the Higgsed five-brane behaves as a string with tension proportional to the separation of the branes. We could consider this heavy string as a probe for the unbroken conformal field theory associated with the N branes that are still together. The procedure is analogous to what we saw above. The Wilson surface operator is defined to be the extra phase factor associated with the interaction of the heavy string with the N five-branes. This Wilson "area" operator in the supergravity picture is defined by requiring that a membrane ends at the boundary of $AdS_7 \times S^4$ on the surface that defines the operator. Notice that we also have to specify a map from the surface to S^4 for the same reasons described above for $\mathcal{N} = 4$ supersymmetric Yang-Mills theory. Again we subtract the term corresponding to the free propagation of the heavy string to obtain a finite result. For large N, we can trust the supergravity result.

As an example, consider a pair of parallel, infinite strings corresponding to membranes ending on the five-brane. Let us choose them with opposite orientation but in the same direction on S^4. This problem is translational invariant along time and the direction of the strings. So the problem of determining the minimal three-surface reduces, as above, to finding the minimum of the action

$$S = \frac{TL'}{(2\pi)^2} \int dx \sqrt{(\partial V)^2 + V^3/R^3}\,, \tag{17}$$

where now $R^3 = \pi N$, and $V = r/l_p^3$ has dimensions of $(\text{energy})^2$. The strings have length L' and are separated by a distance L in the direction $\hat{x}$. We obtain the solution

$$x = \frac{R^{3/2}}{V_0^{1/2}} \int_1^{V/V_0} \frac{dy}{y^{3/2}\sqrt{y^3 - 1}}\,, \tag{18}$$

where $L/2 = x(V = \infty)$. If we calculate the energy, we find

$$\frac{E}{L'} = -\frac{N}{L^2} \frac{8\sqrt{\pi}\,\Gamma(2/3)^3}{\Gamma(1/6)^3}\,. \tag{19}$$

The dependence on L is the one expected from conformal invariance.

Wilson loops in nonconformal theories.—Consider $(2 + 1)$-dimensional supersymmetric Yang-Mills theory with 16 supercharges which is the theory describing coincident D-two-branes. We can define the Wilson loop operator as in (4). Then we are led to consider strings in the background of D-two-branes. The large N limit of this theory was considered in [11], where it was observed that the supergravity description is valid only in some region of the solution. Therefore the analysis of the Wilson loops will also be a bit more involved. We will find that we can calculate the Wilson loops from supergravity only when the size of the loop is not too small. This is just related to the fact that for small distances we can trust the perturbative supersymmetric Yang-Mills theory. The physical result is quite different when the Wilson loop is large. If we consider a string world sheet, embedded in the p-brane solutions studied in [11] in a configuration appropriate for studying the quark-antiquark forces, we find that we have to minimize the action

$$S = \frac{1}{2\pi} \int dx \sqrt{(\partial_x U)^2 + U^5/R^5}\,, \tag{20}$$

where $R^5 = 6\pi^2 g_{\rm YM}^2 N$. We obtain solutions very similar to (10), which lead to the potential

$$E = -\frac{2^{5/3}\sqrt{\pi}\,\Gamma(4/5)^{5/3}}{3^{1/3}\Gamma(3/10)^{5/3}} \frac{(g_{\rm YM}^2 N)^{1/3}}{L^{2/3}} = -\frac{\Gamma(4/5)U_0}{\sqrt{\pi}\,\Gamma(3/10)} \tag{21}$$

between quarks and antiquarks. U_0 is the minimum value of U. Now we perform the analysis of when we can trust (21). Let us first consider the large U region. According to [11], we can trust supergravity for $U \ll g_{\rm YM}^2 N$. The solutions to (20) consist of string world sheets going all the way to $U = \infty$. However, the large U behavior of the solution matches that of the infinitely massive W boson. So we will require the solution at $U \sim g_{\rm YM}^2 N$ to be very similar to that of the W boson; i.e., we require $x - L/2 \ll L$. This implies that $L \gg 1/(g_{\rm YM}^2 N)$. If the distance between the quarks was much smaller than the above bound then we can apply perturbative Yang-Mills and we would obtain a potential proportional to $V \sim g_{\rm YM}^2 N \log(L g_{\rm YM}^2 N)$. We see that these answers match a numerical coefficient with (21) when both calculations break down at $L \sim 1/(g_{\rm YM}^2 N)$.

Now we need to see if we can trust the behavior of the solution at small U, which corresponds to large distances. At small U, we expect that the world sheet of the string turns into an M-two-brane wrapped along the eleventh direction. If $U_0 \gg g_{\rm YM}^2$, then we can trust the above results (21). If U_0 is smaller, then we have to consider a more complicated situation, where we have to solve the equation of the M-two-brane in the background corresponding to a periodic array of M-two-branes as described in [11]: This presumably could be done but we will not attempt to do it here.

I am grateful to the participants of the duality workshop at the Institute for Theoretical Physics at the University of California at Santa Barbara for interesting discussions. I also thank N. Itzhaki for pointing out an error in Eq. (21). This work was supported in part by Grants No. DE-FG02-96ER40559 and No. NSF PHY94-07194. I thank the ITP at UCSB for hospitality.

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