Gravitational Radiation Instability in Hot Young Neutron Stars

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We show that gravitational radiation drives an instability in hot young rapidly rotating neutron stars. This instability occurs primarily in the $l = 2$ r-mode and will carry away most of the angular momentum of a rapidly rotating star by gravitational radiation. On the time scale needed to cool a young neutron star to about $T = 10^9$ K (about one year) this instability can reduce the rotation rate of a rapidly rotating star to about $0.076\Omega_K$, where Ω_K is the Keplerian angular velocity where mass shedding occurs. In older colder neutron stars this instability is suppressed by viscous effects, allowing older stars to be spun up by accretion to larger angular velocities. [S0031-9007(98)06212-7]

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Recently Andersson [1] discovered (and Friedman and Morsink [2] confirmed more generally) that gravitational radiation tends to drive the r-modes of all rotating stars unstable. In this paper we examine the time scales associated with this instability in some detail. We show that gravitational radiation couples to these modes primarily through the current multipoles, rather than the usual mass multipoles. We also evaluate the effects of internal fluid dissipation which tends to suppress this instability. We find that gravitational radiation is stronger than viscosity in these modes and so this instability severely limits the rotation rates of hot young neutron stars. We show that such stars can spin down by the emission of gravitational radiation to about 7.6% of their maximum rotation rates on the time scale (about one year) needed to cool these stars to 10^9 K.

The r-modes of rotating barotropic Newtonian stars are solutions of the perturbed fluid equations having (Eulerian) velocity perturbations

$$\delta\vec{v} = \alpha R\Omega\left(\frac{r}{R}\right)^l \vec{Y}^B_{ll} e^{i\omega t}, \tag{1}$$

where R and Ω are the radius and angular velocity of the unperturbed star, α is an arbitrary constant, and $\vec{Y}^B_{lm}$ is the magnetic-type vector spherical harmonic defined by

$$\vec{Y}^B_{lm} = [l(l+1)]^{-1/2} r\vec{\nabla}\times(r\vec{\nabla}Y_{lm}). \tag{2}$$

Papaloizou and Pringle [3] first showed that the Euler equation for r-modes determines the frequencies as

$$\omega = -\frac{(l-1)(l+2)}{l+1}\Omega. \tag{3}$$

Further use of the Euler equation (as first noted by Provost, Berthomieu, and Rocca [4]) in the barotropic case (a good approximation for neutron stars) determines that only the $l = m$ r-modes exist, and that $\delta\vec{v}$ must have the radial dependence given in Eq. (1). These expressions for the velocity perturbation and frequency are only the lowest order terms in expansions for these quantities in powers of Ω. The exact expressions contain additional terms of order Ω^3.

The lowest order expressions for the (Eulerian) density perturbation $\delta\rho$ can also be deduced from the perturbed fluid equations (Ipser and Lindblom [5]):

$$\frac{\delta\rho}{\rho} = \alpha R^2\Omega^2\frac{d\rho}{dp} \times\left[\frac{2l}{2l+1}\sqrt{\frac{l}{l+1}}\left(\frac{r}{R}\right)^{l+1} + \delta\Psi(r)\right]Y_{l+1\,l}e^{i\omega t}, \tag{4}$$

where $\delta\Psi(r)$ is proportional to the gravitational potential $\delta\Phi$ and satisfies

$$\frac{d^2\delta\Psi}{dr^2} + \frac{2}{r}\frac{d\delta\Psi}{dr} + \left[4\pi G\rho\frac{d\rho}{dp} - \frac{(l+1)(l+2)}{r^2}\right]\delta\Psi = -\frac{8\pi Gl}{2l+1}\sqrt{\frac{l}{l+1}}\,\rho\frac{d\rho}{dp}\left(\frac{r}{R}\right)^{l+1}. \tag{5}$$

Equation (4) is the complete expression for $\delta\rho$ to order Ω^2. The next order terms are proportional to Ω^4.

Our interest here is to study the evolution of these modes due to the dissipative influences of viscosity and gravitational radiation. For this purpose it is useful to consider the effects of radiation on the evolution of the energy of the mode (as measured in the corotating frame of the equilibrium star) $\tilde{E}$:

$$\tilde{E} = \frac{1}{2}\int\left[\rho\,\delta\vec{v}\cdot\delta\vec{v}^* + \left(\frac{\delta p}{\rho} - \delta\Phi\right)\delta\rho^*\right]d^3x. \tag{6}$$

This energy evolves on the secular time scale of the dissipative processes. The general expression for the time derivative of $\tilde{E}$ for a mode with time dependence $e^{i\omega t}$ and azimuthal angular dependence $e^{im\varphi}$ is

$$\frac{d\tilde{E}}{dt} = -\int (2\eta\delta\sigma^{ab}\delta\sigma^*_{ab} + \zeta\delta\sigma\delta\sigma^*)d^3x - \omega(\omega + m\Omega)\sum_{l\geq 2} N_l\omega^{2l}(|\delta D_{lm}|^2 + |\delta J_{lm}|^2). \tag{7}$$

The thermodynamic functions η and ζ that appear in Eq. (7) are the shear and bulk viscosities of the fluid. The viscous forces are driven by the shear $\delta\sigma_{ab}$ and expansion $\delta\sigma$ of the perturbation, defined by the usual expressions

$$\delta\sigma_{ab} = \tfrac{1}{2}(\nabla_a\delta v_b + \nabla_b\delta v_a - \tfrac{2}{3}\delta_{ab}\nabla_c\delta v^c), \tag{8}$$

$$\delta\sigma = \nabla_a\delta v^a. \tag{9}$$

Gravitational radiation couples to the evolution of the mode through the mass δD_{lm} and current δJ_{lm} multipole moments of the perturbed fluid,

$$\delta D_{lm} = \int \delta\rho r^l Y^*_{lm} d^3x, \tag{10}$$

$$\delta J_{lm} = \frac{2}{c}\sqrt{\frac{l}{l+1}}\int r^l(\rho\delta\vec{v} + \delta\rho\vec{v})\cdot\vec{Y}^{B*}_{lm}d^3x, \tag{11}$$

with coupling constant

$$N_l = \frac{4\pi G}{c^{2l+1}}\frac{(l+1)(l+2)}{l(l-1)[(2l+1)!!]^2}. \tag{12}$$

The terms in the expression for $d\tilde{E}/dt$ due to viscosity and the gravitational radiation generated by the mass multipoles are well known [6]. The terms involving the current multipole moments have been deduced from the general expressions given by Thorne [7].

We can now use Eq. (7) to evaluate the stability of the r-modes. Viscosity always tends to decrease the energy $\tilde{E}$, while gravitational radiation may either increase or decrease it. The sum that appears in Eq. (7) is positive definite; thus the effect of gravitational radiation is determined by the sign of $\omega(\omega + m\Omega)$. For r-modes this quantity is negative definite:

$$\omega(\omega + l\Omega) = -\frac{2(l-1)(l+2)}{(l+1)^2}\Omega^2 < 0. \tag{13}$$

Therefore gravitational radiation tends to increase the energy of these modes. For small angular velocities the energy $\tilde{E}$ is positive definite: The positive term $|\delta\vec{v}|^2$ in Eq. (6) (proportional to Ω^2) dominates the indefinite term $(\delta p/\rho - \delta\Phi)\delta\rho^*$ (proportional to Ω^4). Thus, gravitational radiation tends to make *every* r-mode unstable in slowly rotating stars. This confirms the discovery of Andersson [1] and the more general arguments of Friedman and Morsink [2]. To determine whether these modes are actually stable or unstable in rotating neutron stars, therefore, we must evaluate the magnitudes of all the dissipative terms in Eq. (7) and determine which dominates.

Here we estimate the relative importance of these dissipative effects in the small angular velocity limit using the lowest order expressions for the r-mode $\delta\vec{v}$ and $\delta\rho$ given in Eqs. (1) and (4). The lowest order expression for the energy of the mode $\tilde{E}$ is

$$\tilde{E} = \tfrac{1}{2}\alpha^2\Omega^2 R^{-2l+2}\int_0^R \rho r^{2l+2}dr. \tag{14}$$

The lowest order contribution to the gravitational radiation terms in the energy dissipation comes entirely from the current multipole moment δJ_{ll}. This term can be evaluated to lowest order in Ω using Eqs. (1) and (11):

$$\delta J_{ll} = \frac{2\alpha\Omega}{cR^{l-1}}\sqrt{\frac{l}{l+1}}\int_0^R \rho r^{2l+2}dr. \tag{15}$$

The other contributions from gravitational radiation to the dissipation rate are all higher order in Ω. The mass multipole moment contributions are higher order because (a) the density perturbation $\delta\rho$ from Eq. (4) is proportional to Ω^2 while the velocity perturbation $\delta\vec{v}$ is proportional to Ω; and (b) the density perturbation $\delta\rho$ generates gravitational radiation at order $2l+4$ in ω while $\delta\vec{v}$ generates radiation at order $2l+2$.

The contribution of gravitational radiation to the imaginary part of the frequency of the mode $1/\tau_{GR}$ can be computed as follows:

$$\frac{1}{\tau_{GR}} = -\frac{1}{2\tilde{E}}\left(\frac{d\tilde{E}}{dt}\right)_{GR}. \tag{16}$$

Using Eqs. (14)–(16) we obtain an explicit expression for the gravitational radiation time scale associated with the r-modes:

$$\frac{1}{\tau_{GR}} = -\frac{32\pi G\Omega^{2l+2}}{c^{2l+3}} \times \frac{(l-1)^{2l}}{[(2l+1)!!]^2}\left(\frac{l+2}{l+1}\right)^{2l+2}\int_0^R \rho r^{2l+2}dr. \tag{17}$$

The time derivative of the energy due to viscous dissipation is driven by the shear $\delta\sigma_{ab}$ and the expansion $\delta\sigma$ of the velocity perturbation. The shear can be evaluated using Eqs. (1) and (8) and its integral over the constant r two-spheres performed in a straightforward calculation. Using the formulas for the viscous dissipation rate Eq. (7) and the energy Eq. (14), we obtain the contribution of shear viscosity to the imaginary part of the frequency of the mode,

$$\frac{1}{\tau_V} = (l-1)(2l+1)\int_0^R \eta r^{2l}dr\left(\int_0^R \rho r^{2l+2}dr\right)^{-1}. \tag{18}$$

The expansion $\delta\sigma$, which drives the bulk viscosity dissipation in the fluid, can be reexpressed in terms of the density perturbation. The perturbed mass conservation law gives the relationship $\delta\sigma = -i(\omega + m\Omega)\Delta\rho/\rho$, where $\Delta\rho$ is the Lagrangian perturbation in the density. The perturbation analysis used here is not of sufficiently high order (in Ω) to evaluate the lowest order contribution to $\Delta\rho$. However, we are able to evaluate the Eulerian perturbation $\delta\rho$ as given in Eq. (4). We expect that the integral of $|\delta\rho/\rho|^2$ over the interior of the star will be similar to (i.e., within about a factor of 2) the integral of $|\Delta\rho/\rho|^2$. Thus, we estimate the magnitude of the bulk viscosity contribution to the dissipation by

$$\frac{1}{\tau_B} \approx \frac{(\omega + m\Omega)^2}{2\tilde{E}} \int \zeta \, \frac{\delta\rho\,\delta\rho^*}{\rho^2}\, d^3x \,. \tag{19}$$

Using Eqs. (4) and (14) for $\delta\rho/\rho$ and $\tilde{E}$, Eq. (19) becomes an explicit formula for the contribution to the imaginary part of the frequency due to bulk viscosity.

To evaluate the dissipative time scales associated with the r-modes using the formulas in Eqs. (17)–(19), we need models for the structures of neutron stars as well as expressions for the viscosities of neutron star matter. We have evaluated these time scales for $1.4M_\odot$ neutron star models based on several realistic equations of state [8]. We use the standard formulas for the shear and bulk viscosities of hot neutron star matter [9]

$$\eta = 347\rho^{9/4}T^{-2}, \tag{20}$$

$$\zeta = 6.0 \times 10^{-59}\rho^2(\omega + m\Omega)^{-2}T^6, \tag{21}$$

where all quantities are given in cgs units. The time scales for the more realistic equations of state are comparable to those based on a simple polytropic model $p = \kappa\rho^2$ with κ chosen so that the radius of a $1.4M_\odot$ star is 12.53 km. The dissipation time scales for this polytropic model (which can be evaluated analytically) are $\tilde{\tau}_{GR} = -3.26$ s, $\tilde{\tau}_V = 2.52 \times 10^8$ s, and $\tilde{\tau}_B = 6.99 \times 10^8$ s for the fiducial values of the angular velocity $\Omega = \sqrt{\pi G\bar{\rho}}$ and temperature $T = 10^9$ K in the $l = 2$ r-mode. The gravitational radiation time scales increase by about 1 order of magnitude for each incremental increase in l, while the viscous time scales decrease by about 20%.

The evolution of an r-mode due to the dissipative effects of viscosity and gravitational radiation reaction is determined by the imaginary part of the frequency of the mode,

$$\frac{1}{\tau(\Omega)} = \frac{1}{\tilde{\tau}_{GR}}\left(\frac{\Omega^2}{\pi G\bar{\rho}}\right)^{l+1} + \frac{1}{\tilde{\tau}_V}\left(\frac{10^9\text{ K}}{T}\right)^2 + \frac{1}{\tilde{\tau}_B}\left(\frac{T}{10^9\text{ K}}\right)^6\left(\frac{\Omega^2}{\pi G\bar{\rho}}\right). \tag{22}$$

Equation (22) is displayed in a form that makes explicit the angular velocity and temperature dependences of the various terms. Dissipative effects cause the mode to decay exponentially as $e^{-t/\tau}$ (i.e., the mode is stable) as long as $\tau > 0$. From Eqs. (17)–(19) we see that $\tilde{\tau}_V > 0$ and $\tilde{\tau}_B > 0$ while $\tilde{\tau}_{GR} < 0$. Thus gravitational radiation drives these modes towards instability while viscosity tries to stabilize them. For small Ω the gravitational radiation contribution to the imaginary part of the frequency is very small since it is proportional to Ω^{2l+2}. Thus, for sufficiently small angular velocities, viscosity dominates and the mode is stable. For sufficiently large Ω, however, gravitational radiation will dominate and drive the mode unstable. It is convenient to define a critical angular velocity Ω_c, where the sign of the imaginary part of the frequency changes from positive to negative: $1/\tau(\Omega_c) = 0$. If the angular velocity of the star exceeds Ω_c then gravitational radiation reaction dominates viscosity and the mode is unstable.

For a given temperature and mode l the equation for the critical angular velocity, $0 = 1/\tau(\Omega_c)$, is a polynomial of order $l + 1$ in Ω_c^2, and thus each mode has its own critical angular velocity. However, only the smallest of these (always the $l = 2$ r-mode here) represents the critical angular velocity of the star. Figure 1 depicts the critical angular velocity for a range of temperatures relevant for neutron stars. The solid curve in Fig. 1 represents the critical angular velocity for the polytropic model discussed above. Figure 2 depicts the critical angular velocities for $1.4M_\odot$ neutron star models computed from a variety of realistic equations of state [8]. Figure 2 illustrates that the minimum critical angular velocity (in units of $\sqrt{\pi G\bar{\rho}}$) is extremely insensitive to the equation of state. The minima of these curves occur at $T \approx 2 \times 10^9$ K, with $\Omega_c \approx 0.043\sqrt{\pi G\bar{\rho}}$. The maximum angular velocity for any star occurs when the material at the surface effectively orbits the star. This "Keplerian" angular velocity Ω_K is very nearly $\frac{2}{3}\sqrt{\pi G\bar{\rho}}$ for any equation of state. Thus the minimum critical angular velocity due to instability of the r-modes is about $0.065\Omega_K$ for any equation of state [10].

To determine how rapidly a young neutron star is allowed to spin after cooling, we must compare the rate it cools with the rate it loses angular momentum

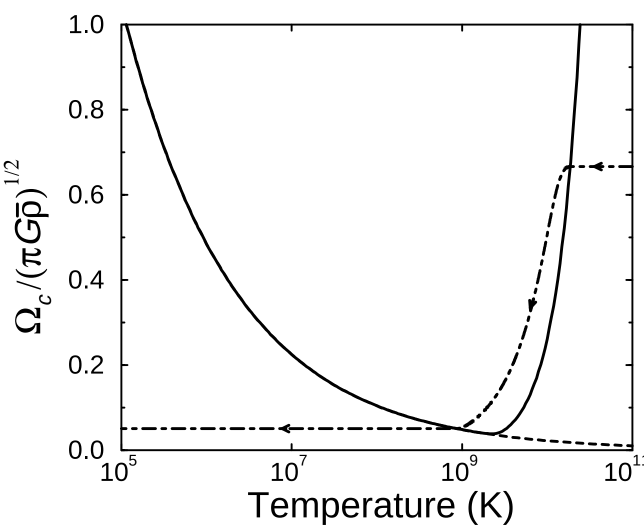

FIG. 1. Critical angular velocities for a $1.4M_\odot$ polytropic neutron star with (solid line) and without (dashed line) bulk viscosity. Also the evolution of a rapidly rotating neutron star (dash-dotted line) as the star cools and emits gravitational radiation.

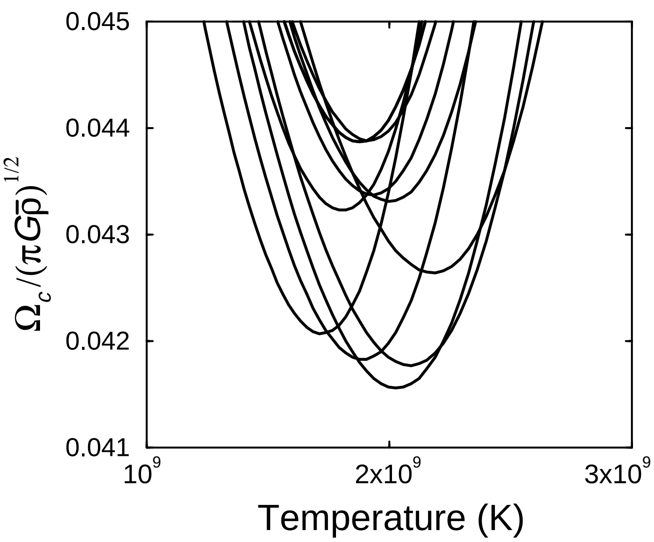

FIG. 2. Critical angular velocities of realistic $1.4M_\odot$ neutron star models.

by gravitational radiation. We approximate the cooling with a simple model based on the emission of neutrinos through the modified URCA process [11]. We compute the time evolution of the angular velocity of the star by setting $dJ/dt = J/\tau$, where J is the angular momentum of the star and τ is the time scale given in Eq. (22). The result is a simple first order differential equation for $\Omega(t)$ which we solve for initial angular velocity $\Omega = \Omega_K$ and initial temperature 10^{11} K. The solution is shown as the dash-dotted line in Fig. 1. The gravitational radiation time scale is so short that the star radiates away its angular momentum almost as quickly as it cools. The angular velocity of the star decreases from Ω_K to $0.076\Omega_K$ in a period of about one year [12]. Thus, we conclude that young neutron stars will be spun down by the emission of gravitational radiation within their first year to a rotation period of about $13P_{\min}$, where $P_{\min} = 2\pi/\Omega_K$. The Crab pulsar with present rotation period 33 ms and initial period 19 ms (based on the measured braking index) rotates more slowly than this limit if $P_{\min} < 1.5$ ms.

Our analysis is based on the assumption that a young hot neutron star may be modeled as an ordinary fluid. Once the star cools below the superfluid transition temperature (about 10^9 K) the analysis presented here must be modified [13]. We expect the r-mode instability to be completely suppressed (with $\Omega_c = \Omega_K$) when the star becomes a superfluid [14]. This makes it possible for old recycled pulsars to be spun up to large angular velocities by accretion if they are not reheated above 10^9 K in the process. If nonperfect fluid effects enter above 10^9 K, however, the spin-down process may be terminated at a higher angular velocity than the $0.076\Omega_K$ figure computed here. The detection of a young fast pulsar [15] would provide evidence for such effects at temperatures higher than 10^9 K. Magnetic fields could also damp these modes; however preliminary estimates based on standard magnetosphere-mode coupling models [16] suggest that such damping is too weak to suppress the relatively low frequency r-mode instability.

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