Noise-Induced Hypersensitivity to Small Time-Dependent Signals

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For a simple example of on-off intermittency, an overdamped Kramers oscillator with multiplicative noise, we demonstrate a phenomenon of hypersensitivity to ultrasmall time-dependent signals. [S0031-9007(98)06229-2]

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The phenomenon of on-off intermittency introduced recently [1–3] attracts now a growing interest of investigators in various natural sciences. The key feature of the systems with this type of intermittency is the large magnitude of fluctuations of physical variables, which can take both finite and extremely small (in laminar phase) values with comparable probabilities. In the present Letter we describe a remarkable phenomenon that we found in such systems, namely, an immense response of the system to an ultrasmall external perturbation (hypersensitivity), when the signal value, e.g., of the order of 10^{-20} results in response value of the order of unity. We demonstrate such a hypersensitivity for one of the simplest examples of on-off intermittency, the overdamped Kramers oscillator with multiplicative noise.

The equations that describe our model are the following:

$$\frac{dx}{dt} = \lambda x + \beta\xi(t)x - Ux^3 + \sigma\varphi(t) + AR(t),$$
$$\langle\xi(t)\xi(t')\rangle = \langle\varphi(t)\varphi(t')\rangle = \delta(t - t'),$$
$$\langle\xi(t)\varphi(t')\rangle = 0, \tag{1}$$
$$R(t + T) = R(t) = \begin{cases} 1, & 0 < t \leq \frac{T}{2}, \\ -1, & \frac{T}{2} < t \leq T. \end{cases}$$

Here $\xi(t), \varphi(t)$ are the Gaussian white noise sources, $\lambda, \beta, U, \sigma, A$ are the constant parameters, and $R(t)$ is the periodic input signal. Equation (1) is written in Stratonovich sense. The case $A = \sigma = 0$ was studied in detail in [4]. Equation (1) contains both additive and multiplicative noise terms, and just the latter is responsible for hypersensitivity, as shown below.

In the noise-free case ($\beta = \sigma = 0$) for $A \ll 1$, Eq. (1) can be solved easily, and one could see that the output signal amplitude $\Delta x \sim A/|\lambda|$; i.e., the system does not amplify signal at all. The Fokker-Planck equation (FPE) for Eq. (1) has the form

$$\frac{\partial F}{\partial t} = -\frac{\partial}{\partial x}\left\{\left[\left(\lambda + \frac{\beta^2}{2}\right)x - Ux^3 + AR(t)\right]F\right\} + \frac{1}{2}\frac{\partial^2}{\partial x^2}\{(\beta^2 x^2 + \sigma^2)F\}. \tag{2}$$

In general, it is a rather complicated task to solve Eq. (2) exactly, so we use the following approximation. The signal $R(t)$ in (1) takes two values ± 1. Let T_0 be the time for establishing an equilibrium after switching the signal from one of these values to another. We assume that the signal satisfies the adiabatic condition

$$T \gg T_0. \tag{3}$$

Solving FPE in this case, we obtain for $(A, \sigma) \ll (\lambda, \beta, U)$

$$F(x) = C\left(x^2 + \frac{\sigma^2}{\beta^2}\right)^{(\alpha-1)/2} \exp\left\{\frac{2AR(t)}{\beta\sigma} \arctan\frac{\beta x}{\sigma} - \frac{Ux^2}{\beta^2}\right\}, \quad \alpha = \frac{2\lambda}{\beta^2}, \tag{4}$$

where C is the normalization constant.

Our working range of parameters is $\beta, U \sim 1, \lambda \sim 0.01, A, \sigma \sim 10^{-n}, n \gg 1$. Then we obtain from Eq. (4) the probability density of scaling type in a wide interval $10^{-n} \ll x \ll 1$,

$$F(x) = |x|^{\alpha-1}. \tag{5}$$

The power-law distribution of x is known as one of the signatures of on-off intermittency [5–7].

We restrict ourselves by the limit $\sigma \to 0$ (the small signal is much greater than the additive noise). Taking in mind that $\arctan\frac{\beta x}{\sigma} \overset{\sigma\to 0}{\longrightarrow} \frac{\pi}{2}\operatorname{sgn} x - \frac{\sigma}{\beta x}$, we obtain

$$F(x) = C|x|^{\alpha-1}\exp\left\{\frac{A\pi R(t)}{\beta\sigma}\operatorname{sgn} x - \frac{2AR(t)}{\beta^2 x} - \frac{Ux^2}{\beta^2}\right\}. \tag{6}$$

The first term in the exponent means that for positive signal the probability density is nonzero only when x is positive, and for negative $R(t)$, correspondingly, when x is negative. Thus Eq. (6) takes the form

$$F(x) = C|x|^{\alpha-1}\theta[\operatorname{sgn} AR(t)x]\exp\left\{-\frac{2AR(t)}{\beta^2 x} - \frac{Ux^2}{\beta^2}\right\}, \tag{7}$$

where $\Theta(x)$ is the Heaviside step function.

The normalization constant C cannot be obtained in exact form, but its asymptotics for $|\alpha| \ll 1, \frac{U}{\beta^2} \sim 1$ are the following:

$$C = \begin{cases} \alpha; & \alpha > 0,\ z \gg 1\,, \\ \frac{1}{\ln\frac{1}{A}}; & z \ll 1\,, \\ |\alpha|A^{|\alpha|}; & \alpha < 0,\ z \gg 1\,, \end{cases} \tag{8}$$

$$z = |\alpha| \ln \frac{1}{A}\,.$$

The crossover of these asymptotics takes place when the parameter z is of the order of unity, i.e., when the signal amplitude is

$$A_0 = \exp\left(-\frac{1}{|\alpha|}\right). \tag{9}$$

Therefore, for small α an ultrasmall signal is able to change the probability density drastically. To obtain an estimation of output signal amplitude, let us calculate the moments of $F(x)$ for $z \ll 1$. Taking into account the explicit form of $R(t)$, we obtain

$$\begin{aligned} \langle x(t)\rangle &= \frac{\beta}{2}\sqrt{\frac{\pi}{U}}\,\frac{1}{\ln\frac{1}{A}}\,R(t)\,, \\ \langle x^2(t)\rangle &= \frac{\beta^2}{2U}\,\frac{1}{\ln\frac{1}{A}}\,, \\ \frac{\langle x(t)\rangle^2}{\langle x^2(t)\rangle} &= \frac{\pi}{2\ln\frac{1}{A}} \ll 1\,. \end{aligned} \tag{10}$$

The gain factor is

$$I = \frac{\langle x(t)\rangle}{AR(t)} = \sqrt{\frac{\pi}{4U}}\,\frac{\beta}{A\ln\frac{1}{A}}\,. \tag{11}$$

As an example, for $\beta = 0.7, U = 1, A = 10^{-11}$ the value of I is 2.5×10^9.

We see that our simple model has an amazing feature of hypersensitivity to small signals due to multiplicative noise. We should note that the related problem of additive noise-induced sensitivity to an ultrasmall *static* perturbation was studied by Kondepudi *et al.* (see [8] and references therein) in the context of branch selection in chemical system. In our system the hypersensitivity is induced by multiplicative noise and, much more important, the signal can be *time dependent.* Despite relatively large dispersion of x, the phenomenon is observable easily with the help of usual statistical methods. Figure 1 displays the normalized time series $\langle x(t)\rangle$ for $A = 10^{-11}, \lambda = -0.01$ and $0.01, \beta = 1.0, U = 1$ as an average on 4100 runs of the model with the same phase of input signal. Note that, despite relatively small *averaged* output values, the *instant* output signal values can be of the order of unity, thus allowing one to detect ultrasmall signal with a low sensitive detector. When the adiabatic condition is fulfilled, we see from Eq. (10) that $\langle x(t)\rangle = yR(t)$, and the gain factor $I = y/A$. When the signal is nonadiabatic, we can define I as

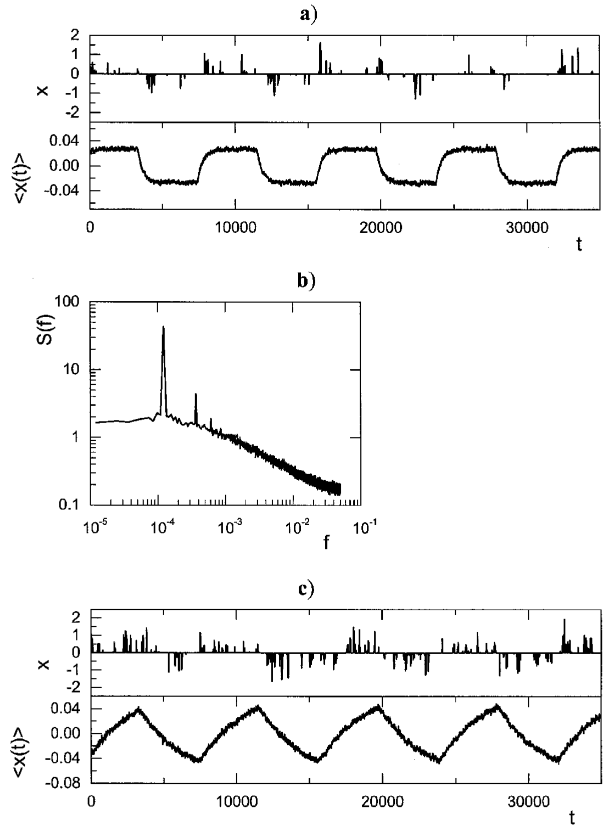

FIG. 1. (a) Time series of raw output signal x and its ensemble average $\langle x(t)\rangle$ over 4100 samples with identical phase of input signal for $\beta = 1.0, \lambda = -0.01$; (b) power spectrum of x for the signal (a), averaged over 200 samples of random phase; (c) the same as for (a) but for $\lambda = 0.03$. The input signal amplitude $A = 10^{-11}$, period $T = 8192$.

$$I^2 = \frac{1}{T}\int_0^T \frac{\langle x(t)\rangle^2}{A^2}\,dt\,. \tag{12}$$

From Fig. 1 it is seen that even after an averaging on 4100 samples $\langle x(t)\rangle$ still fluctuates noticeably, and it is more convenient to calculate the factor I in the following way. It was shown in [9] that, when we have an ensemble of time series with random phase lag, their spectral density is $S(\omega) = 2\pi\sum_k |x_k|^2\delta(\omega - k\Omega) + S_{\text{noise}}(\omega)$, where x_k are the Fourier coefficients of periodic function $\langle x(t)\rangle$, and S_{noise} is the purely stochastic term. From Eq. (12) we see that $I^2 = \sum_k |x_k|^2/A^2$, i.e., $I^2 = \delta f \sum_i (S_i - S_{\text{noise}})/A^2$, where S_i is the ith harmonic in the spectrum and δf is the spectral bandwidth. The dependence of gain factor I on parameters λ and α for fixed noise value $\beta = 0.7$ is shown in Fig. 2(a). We see that the estimate (11) (which is written for $|\alpha| \approx 0$) is in excellent agreement with

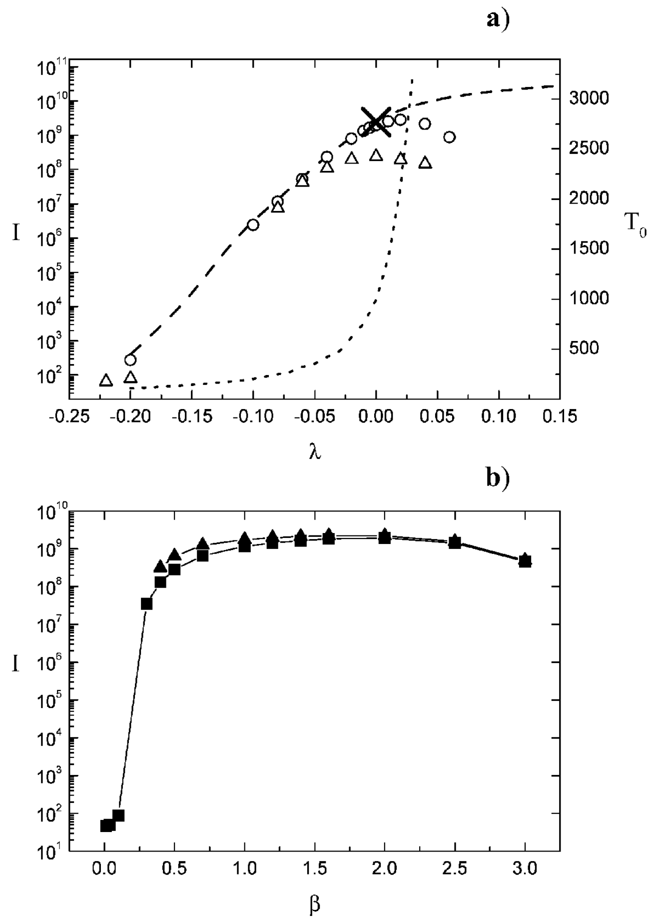

FIG. 2. (a) Dependence of gain factor I for signal period $T = 819$ (triangles), $T = 8192$ (circles), and relaxation time T_0 (dotted line) on parameter λ for fixed $\beta = 0.7$. Dashed line represents simulations with a constant input signal. The cross displays an estimate (11). (b) The gain factor I versus noise intensity β for values of $\lambda = -0.01$ (squares) and 0.01 (triangles). The input signal is the same as in Fig. 1.

simulation results. The dashed line represents the case of static signal $[R(t) = 1]$. We see also that the gain factor decreases from its static value when the relaxation time T_0 [its values obtained from simulation are shown by the dotted line in Fig. 2(a) and the theoretical estimate is $T_0 \sim A^{-\alpha}$] becomes comparable to the signal period, i.e., the adiabatic condition is no longer valid. When we increase the signal period, the range of adiabaticity obviously broadens. Figure 2(b) demonstrates the dependence of the gain factor on the amplitude of multiplicative noise (this latter might be regarded as a control parameter in the model). This bell-shaped dependence resembles very much a conventional stochastic resonance, with the difference that in our system the signal is additive and the noise is multiplicative.

To conclude, we demonstrate, both analytically and by computer simulations, that an overdamped Kramers oscillator with multiplicative noise, the simple stochastic system with on-off intermittency, for small values of parameter α possess the feature of *noise-induced hypersensitivity to small time-dependent signals.* Such a sensitivity appears when the distribution of x obeys a power-law dependence in a wide interval $10^{-n} \ll x \ll 1$, and thus one might expect to observe it in any system with on-off intermittency.

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