High-Temperature Series Analysis of 2D Random-Bond Ising Ferromagnets

Alexandra Roder,[1] Joan Adler,[2] and Wolfhard Janke[1,3]
[1]*Institut für Physik, Johannes Gutenberg-Universität Mainz, Staudinger Weg 7, 55099 Mainz, Germany*
[2]*Department of Physics, Technion, Haifa 32000, Israel*
[3]*Institut für Theoretische Physik, Universität Leipzig, Augustusplatz 10/11, 04109 Leipzig, Germany*
(Received 25 February 1998)

We analyze new high-temperature series expansions for the susceptibility χ of the two-dimensional random-bond Ising ferromagnet with a symmetric bimodal distribution of two positive couplings J_1 and J_2. By studying a wide range of coupling ratios J_2/J_1 we obtain compelling evidence for a critical behavior of the form $\chi \sim t^{-7/4}|\ln t|^{7/8}$, as predicted theoretically by Shalaev, Shankar, and Ludwig. [S0031-9007(98)06162-6]

PACS numbers: 64.60.Fr, 64.60.Ak, 75.10.Jm

The study of critical phenomena in real materials is often complicated by the influence of impurities and inhomogeneities. Since this is the typical situation in most experiments, it is of great importance to develop a theoretical understanding of the role of these nonideal effects. One important question is whether the critical properties of the considered system are robust against such random perturbations and, if not, by how much they are modified. In many applications the dynamics of the impurities is much slower than the dynamics of the other degrees of freedom. Then one may treat the impurities as frozen disorder, i.e., consider the "quenched" approximation. An important guideline for the importance of quenched, random disorder is the Harris criterion [1] which states that a perturbation is relevant when the critical exponent α of the specific heat of the pure system is positive. In this case the critical properties will be modified by the influence of quenched disorder. Pure systems with a negative α, on the other hand, are not expected to change their critical behavior in the presence of quenched disorder. The marginal case is $\alpha = 0$, where the Harris criterion cannot make any prediction.

The paradigm of this marginal situation is the two-dimensional Ising model subject to quenched, random bond-disorder [2]. Because of its relative simplicity and the fact that $\alpha = 0$ in the pure case is known exactly, this model has been the subject of many theoretical investigations [3–7], numerical Monte Carlo simulations [8–13], and transfer matrix studies [14,15]. Despite all these efforts, however, theoretical controversies about the critical behavior of the susceptibility could never really be resolved. This motivated us to study this problem yet again using an independent approach, namely high-temperature series expansions.

The random-bond Ising ferromagnet is defined by the Hamiltonian:

$$\mathcal{H} = -\sum_{\langle ij\rangle} J_{ij}\sigma_i\sigma_j\,, \qquad (1)$$

where the spins $\sigma_i = \pm 1$ are located at the sites of a square lattice of size V and the symbol $\langle ij\rangle$ denotes nearest-neighbor interactions. The coupling constants J_{ij} are quenched, random variables which are drawn from a bimodal distribution,

$$P(J_{ij}) = x\delta(J_{ij} - J_1) + (1 - x)\delta(J_{ij} - J_2)\,, \qquad (2)$$

of two positive couplings J_1 and J_2. In the symmetric case, $x = 1/2$, the critical temperature T_c of the model is known exactly [2]. In the high-temperature phase the magnetic susceptibility per site is defined as

$$\chi = \lim_{V\to\infty}\left[\left\langle\left(\sum_{i=1}^{V}\sigma_i\right)^2\right\rangle_{\mathrm{T}} \Big/ V\right]_{\mathrm{av}}, \qquad (3)$$

where $\langle\cdots\rangle_{\mathrm{T}}$ denotes the usual thermal average with respect to $\exp(-\mathcal{H}/k_BT)$, and the bracket $[\cdots]_{\mathrm{av}}$ denotes the average over the quenched, random disorder.

There are two competing theoretical predictions. Based on renormalization-group techniques, Dotsenko and Dotsenko (DD) [3] predicted a critical behavior of the form

$$\chi \sim t^{-2}\exp\left[-a\left(\ln\ln\left(\frac{1}{t}\right)\right)^2\right], \qquad (4)$$

where $t = (T - T_c)/T_c \geq 0$ denotes the reduced temperature, and a is a constant which depends on the strength of the disorder. A few years later this prediction was questioned by Shalaev, Shankar, and Ludwig (SSL) [4–7] who used bosonization techniques and the method of conformal invariance to derive quite a different behavior,

$$\chi \sim t^{-7/4}|\ln t|^{7/8}. \qquad (5)$$

This is the same leading singularity as in the pure case, but modified by a multiplicative logarithmic correction.

Most high-precision Monte Carlo simulations and transfer matrix studies [8–14] favor the latter form, but could not confirm the multiplicative logarithmic correction in (5) quantitatively. Therefore, we found it worthwhile to investigate this problem once again with the independent method of high-temperature series expansions. To derive the series expansions of the susceptibility (3) up to the 11th order in $k = 2J_1/k_BT$ we adapted an algebraic computer program package developed originally for the

d-dimensional q-state Potts spin-glass model (where $J_{ij} = \pm J$) [16–19]. Even though spin-glass and random-bond systems are physically very different [20], precisely the same enumeration scheme for the high-temperature graphs can be employed. The only difference is in the last step where the quenched averages over the J_{ij} are performed. The details of the star-graph expansion technique and our specific implementation are described elsewhere [16–19,21]. In this Letter we shall concentrate on the susceptibility series for the two-dimensional ($d = 2$) random-bond Ising model ($q = 2$) with a symmetric ($x = 1/2$) bimodal distribution of two positive coupling strengths J_1 and J_2 (the implementation, however, is completely general).

For the analysis of the series we employed two different methods. The first is based on the ratio method [22,23], which for a generic thermodynamic function

$$F(z) = \sum_{n=0} a_n z^n \sim (1 - z/z_c)^{-\lambda} \tag{6}$$

amounts to computing the ratios

$$r_n \equiv \frac{a_n}{a_{n-1}} \sim \left[1 + \frac{\lambda - 1}{n}\right]\frac{1}{z_c}, \tag{7}$$

and extracting the critical point z_c and the critical exponent λ from the offset and slope of this sequence as a function of $1/n$, respectively. If the critical point z_c is known from other sources (in our case exactly from self-duality), then one may consider biased extrapolants for the critical exponent, $\lambda_n = n r_n z_c - n + 1$, which simply follow by rearranging Eq. (7). In the following this method will be denoted as the "biased ratio I."

If the singularity of $F(z)$ contains a multiplicative logarithmic correction [as, e.g., in the SSL prediction (5)],

$$F(z) \sim (1 - z/z_c)^{-\lambda}|\ln(1 - z/z_c)|^p, \tag{8}$$

then one forms the ratios r_n as before, but considers in addition the auxiliary function [24]

$$z^{-p^*}(1 - z)^{-\lambda}\{\ln[1/(1 - z)]\}^{p^*} = \sum_{n=0}^{N} b_n z^n + \ldots, \tag{9}$$

and computes the ratios $r_n^* \equiv b_n/b_{n-1}$. If λ is known, it can be shown that the sequence $R_n \equiv r_n/r_n^*$ approaches $1/z_c$ with zero slope in the limit $n \to \infty$, if and only if $p^* = p$. This determines p, if also z_c is known. If λ is not known, then one may vary both exponents until the above relation is satisfied. In the following, we refer to this special ratio method as the "ln-ratio."

The second method [25,26] suitable for a singularity of the form (8) is based on Padé approximants [27]. Here one constructs the auxiliary function

$$G(z) = -(z_c - z)\ln(z_c - z)\left(\frac{F'(z)}{F(z)} - \frac{\lambda}{z_c - z}\right), \tag{10}$$

which satisfies $\lim_{z\to z_c} G(z) = p$. If z_c is known, the value of $G(z)$ at z_c can be obtained by computing standard Padé approximants to $G(z)$,

$$G(z) \approx [L/M] \equiv \frac{P_L(z)}{Q_M(z)} \equiv \frac{p_0 + p_1 z + \cdots + p_L z^L}{1 + q_1 z + \cdots + q_M z^M}, \tag{11}$$

where $L + M \le N - 1$. We therefore call this method "ln-Padé."

Our first step was to investigate whether the susceptibility series are consistent with a pure power-law behavior according to the DD prediction (4) (ignoring the exponentially small multiplicative correction term). Assuming thus the behavior $\chi \sim t^{-\gamma}$ and using the biased ratio I method we obtained the critical exponents γ shown in Fig. 1 as a function of J_2/J_1. Here and in the following the error bars are estimated by varying the length of the series and/or the type of Padé approximants used. Starting with $\gamma = 1.738 \pm 0.014$ for the pure case ($J_2/J_1 = 1$), being consistent with the exact value of $\gamma = 7/4$, we observe a steady increase to $\gamma = 2.37 \pm 0.11$ for the strongest investigated disorder ($J_2/J_1 = 10$). We will argue below that the apparent crossover from weak to strong disorder is mainly due to the finite length of our series expansion which naturally has a much more dramatic influence for weak disorder. At any rate, for strong disorder the DD prediction of $\gamma = 2$ is clearly outside the error margins of the series analysis estimates.

So far no multiplicative logarithmic corrections were taken into account. If the SSL prediction (5) was correct we would, therefore, expect to observe "effective" critical exponents which according to

$$\chi \sim t^{-7/4}|\ln t|^{7/8} \sim t^{-(7/4)[1+\frac{1}{2}\frac{\ln(|\ln t|)}{\ln(1/t)}]} \tag{12}$$

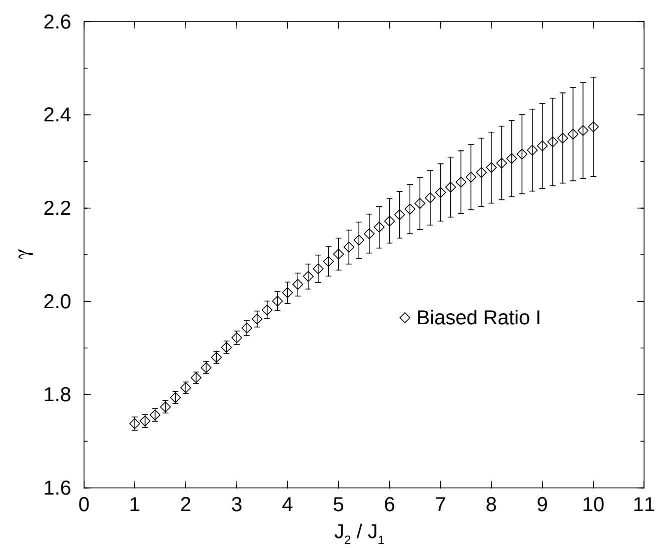

FIG. 1. Analysis of the susceptibility series assuming a singularity of the form $\chi \sim t^{-\gamma}$, using the biased ratio I method.

should indeed be larger than $7/4$. The results in Fig. 1 could thus be well consistent with a critical exponent of $\gamma = 7/4$ in the presence of a multiplicative logarithmic correction.

This possibility suggested a more careful analysis based on the qualitative form of the SSL prediction (5). Our series are too short to employ a general ansatz with both exponents as free parameters. We rather fixed the exponent of the leading term to the (predicted) pure Ising model value and enquired if our series expansions are compatible with the ansatz

$$\chi \sim t^{-7/4} |\ln t|^p, \tag{13}$$

and $p = 7/8$. Employing the two special methods for this type of singularity described above we obtained well converging results. The resulting estimates for the exponent p are shown in Fig. 2. We see that the two methods yield consistent results which start in the pure case ($J_2/J_1 = 1$) around $p = 0$, as they should do. With increasing disorder the estimates exhibit again an apparent crossover, until around $J_2/J_1 = 5 - 8$ they settle at a plateau value in very good agreement with the theoretical prediction of $p = 7/8$. This is the main result of our series analysis which for the first time yields a clear quantitative confirmation of the SSL prediction (5).

As before, we attribute the apparent crossover for intermediate strength of the disorder to the shortness of our series expansions, i.e., we interpret the crossover as an unavoidable artifact of high-temperature series expansion analyses and not as an indication that the exponent p really is a function of the disorder strength. We thus take the view that already a small amount of disorder drives the system into a new universality class different from the pure case which, however, only becomes visible in the close vicinity of the transition point T_c (or $t = 0$). This in turn translates into the need of extremely long series expansions in order to be detectable.

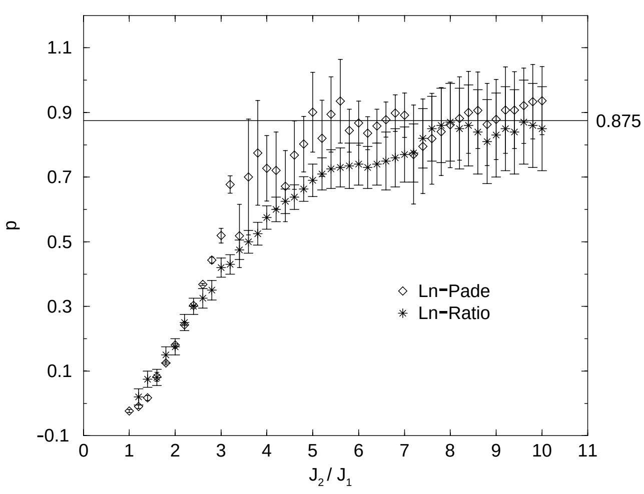

FIG. 2. Analysis of the susceptibility series assuming a singularity of the form $\chi \sim t^{-7/4}|\ln t|^p$, using Padé approximants and the ratio method (see text). The horizontal line at $p = 7/8 = 0.875$ is the theoretical prediction of SSL.

To justify this claim we have investigated a model function simulating the "true" susceptibility ($g_0 \geq 0$),

$$\chi_{\text{model}} = \dot{t}^{-7/4} \left[1 + \frac{4g_0}{\pi} \ln(1/\dot{t}) \right]^{7/8}, \tag{14}$$

with $\dot{t} = (T - T_c)/T$, which for any $g_0 \neq 0$ reproduces the SSL form (5) in the limit $T \to T_c$ ($\dot{t} \approx t \to 0$). Notice the discontinuity in the asymptotic behavior at $g_0 = 0$. For weak disorder g_0 is very small and hence the asymptotic region in t is extremely narrow. By using the weak disorder results the parameter g_0 can be related at least heuristically to the ratio J_2/J_1. A typical value is $g_0 = 0.013$ for $J_2/J_1 = 1.2$. This shows that for weak disorder the asymptotic region is bounded by $t \ll \exp(-\pi/4g_0) \approx \exp(-1/0.017) \approx 10^{-26}$. When the model series was truncated at a low order we observed precisely the same crossover effect as for the "true" series. Based on this model function we are sure, however, that this must be an artifact of the truncation of the model series at a finite order.

To summarize the above results: we have obtained, directly for strong disorder (large J_2/J_1) and by argument from comparison with model series for weaker disorder, compelling evidence that the singularity of the susceptibility is properly described by $\chi \sim t^{-7/4}|\ln t|^p$, with $p = 7/8 = 0.875$, as theoretically predicted by SSL [4–7]. The analysis of the model susceptibility (14) clearly showed that the apparent variation of p with the strength of disorder is an artifact caused by the truncation of the series expansions at a finite order. We emphasize that the apparent crossover from weak to strong disorder does *not* imply that the universality class of the random-bond Ising model changes continuously with the strength of disorder. We suspect that a similar artifact occurs in the finite-size scaling analysis of a related problem [28] and that their change of γ is also spurious. Our measurement centered on $p = 0.875$ over the range $5 \leq J_2/J_1 \leq 10$ is most unlikely to be coincidental and hence validates both SSL and fixed γ decisively.

Our confirmation of the SSL exponents is direct, quantitative, and conclusive, and we conclude this Letter with a discussion of why the nature of the series method enables it to be so much stronger than that of previous Monte Carlo simulations of this model. The first reason for ambivalence from simulation results occurred in the finite-size scaling analyses of Refs. [8–11,13,14], where it is conceptually impossible to detect the multiplicative logarithmic correction of the SSL prediction (5). The reason is that the SSL theory also predicts a logarithmic correction for the scaling behavior of the correlation length, $\xi \sim t^{-1}|\ln t|^{1/2}$. In the finite-size scaling behavior the two logarithms cancel and one ends up with a pure power-law, $\chi \sim L^{\gamma/\nu} = L^{7/4}$, where L is the linear lattice size. Thus *only* the SSL prediction for

γ/ν can be tested in finite-size scaling analyses. Wang *et al.* [9,10] obtained for $J_2/J_1 = 4$ and 10 an estimate of $\gamma/\nu = 1.7507 \pm 0.0014$, and also the results of Reis *et al.* [14] at $J_2/J_1 = 2$, 4, and 10 are consistent with 1.75. While among the two alternatives, the theories of DD and SSL, this may be interpreted as an evidence in favor of SSL, a numerical estimate of $\gamma/\nu \approx 1.75$ would also be expected for the *pure* Ising model and is therefore not really conclusive. To overcome this limitation Talapov and Shchur [12] simulated the temperature dependence of χ for $J_2/J_1 = 4$ directly. From least-squares fits to a power law $\chi \sim t^{-\gamma}$, they obtained an effective exponent $\gamma \approx 7/4 + 0.135 = 1.885$. Notice that this value is quite close to our series estimate of $\gamma = 2.019 \pm 0.024$ for $J_2/J_1 = 4$, if the pure power-law ansatz is used (cp. Fig. 1). Even though this points in the right direction if the SSL prediction is correct, it is only a measurement at a single J_2/J_1 value, thus it is still fair to conclude that also this simulation has not unambiguously identified the multiplicative logarithmic correction term.

With series expansions there is no problem scanning many parameter choices. Series expansions are closed expressions in several parameters (such as the dimension $d, x, J_2/J_1, \ldots$) up to a certain order in the inverse temperature $1/k_BT$. Here the infinite-volume limit is always implied and the quenched, random disorder can be treated exactly. There are no problems with either sample equilibration or random number choice. Scanning is severely limited in Monte Carlo simulations of systems with quenched, random disorder which require an enormous amount of computing time because many realizations of the disorder have to be simulated for the quenched average. It is this scanning which enabled us to find the plateau in Fig. 2. Locating this plateau, combined with the possibility of measuring γ directly, led to our conclusive results. (Another example of the power of the scanning combined with direct exponent measurement within the series method was provided by [29] where confirmation of two-exponent scaling for the random-field Ising model was found.)

We would suggest that independent confirmation from simulations could now be obtained by carrying out further calculations of the Talapov and Shchur type at two different J_2/J_1 values that are well in the plateau of Fig. 2. A longer manuscript, with details of the generation of the series and an analysis of series for the specific heat is in preparation. Our preliminary specific-heat analyses support the above picture.

We wish to thank Professor Kurt Binder for many helpful discussions and his constant interest in this project. W. J. acknowledges support from the Deutsche Forschungsgemeinschaft (DFG) within the Heisenberg program. Partial support of J. A. and W. J. from the German-Israel-Foundation (GIF) under Contract No. I-0438-145.07/95 is also gratefully acknowledged. We thank Professor Dietrich Stauffer for comments on the manuscript.

[1] A. B. Harris, J. Phys. C **7**, 1671 (1974).
[2] F. Y. Wu, Rev. Mod. Phys. **54**, 235 (1982); **55**, 315(E) (1983).
[3] Vik. S. Dotsenko and Vl. S. Dotsenko, JETP Lett. **33**, 37 (1981); Adv. Phys. **32**, 129 (1983).
[4] B. N. Shalaev, Sov. Phys. Solid State **26**, 1811 (1984).
[5] B. N. Shalaev, Phys. Rep. **237**, 129 (1994).
[6] R. Shankar, Phys. Rev. Lett. **58**, 2466 (1987); **61**, 2390 (1988).
[7] A. W. W. Ludwig, Phys. Rev. Lett. **61**, 2388 (1988); Nucl. Phys. **B330**, 639 (1990).
[8] V. B. Andreichenko, Vl. S. Dotsenko, W. Selke, and J.-S. Wang, Nucl. Phys. **B344**, 531 (1990).
[9] J.-S. Wang, W. Selke, Vl. S. Dotsenko, and V. B. Andreichenko, Europhys. Lett. **11**, 301 (1990).
[10] J.-S. Wang, W. Selke, Vl. S. Dotsenko, and V. B. Andreichenko, Physica (Amsterdam) **164A**, 221 (1990).
[11] W. Selke, L. N. Shchur, and A. L. Talapov, in *Annual Reviews of Computational Physics I,* edited by D. Stauffer (World Scientific, Singapore, 1994), p. 17.
[12] A. L. Talapov and L. N. Shchur, J. Phys. Condens. Matter **6**, 8295 (1994).
[13] S. Wiseman and E. Domany, Phys. Rev. E **51**, 3074 (1995); **52**, 3469 (1995).
[14] F. D. A. Aarão Reis, S. L. A. de Queiroz, and R. R. dos Santos, Phys. Rev. B **56**, 6013 (1997).
[15] D. Stauffer, F. D. A. Aarão Reis, S. L. A. de Queiroz, and R. R. dos Santos, Int. J. Mod. Phys. **C8**, 1209 (1997).
[16] G. Schreider, Ph.D. thesis, Universität Mainz, 1993 (unpublished).
[17] G. Schreider and J. D. Reger, J. Phys. A **27**, 1071 (1994); **27**, 1079 (1994).
[18] B. Lobe, Ph.D. thesis, Universität Mainz, 1997 (unpublished).
[19] B. Lobe and W. Janke (to be published).
[20] K. Binder and J. D. Reger, Adv. Phys. **41**, 547 (1992).
[21] A. Roder, diploma thesis, Universität Mainz, 1997 (unpublished).
[22] A. J. Guttmann, in *Phase Transitions and Critical Phenomena,* edited by C. Domb and J. L. Lebowitz (Academic Press, New York, 1989), Vol. 13, p. 1.
[23] D. L. Hunter and G. A. Baker, Jr., Phys. Rev. B **7**, 3346 (1973).
[24] A. J. Guttmann, J. Phys. **A11**, L103 (1978).
[25] J. Adler and V. Privman, J. Phys. A **14**, L463 (1981).
[26] J. Adler and I. Chang, Int. J. Mod. Phys. C **8**, 567 (1997).
[27] G. A. Baker, Jr., *Essentials of Padé Approximants* (Academic Press, London, 1975).
[28] J.-K. Kim and A. Patrascioiu, Phys. Rev. Lett. **72**, 2785 (1994).
[29] M. Gofman, J. Adler, A. Aharony, A. B. Harris, and M. Schwartz, Phys. Rev. Lett. **71**, 1569 (1993).