# Localization in Discontinuous Quantum Systems

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Classical and quantum properties of a discontinuous perturbed twist map are investigated. Different classical diffusive regimes, quasilinear and slow, respectively, are observed. The regime of slow classical diffusion gives rise to two distinct quantal regimes, one marked by dynamical localization, the other by quasi-integrable localization due to classical cantori. In both cases the resulting quantum stationary distributions are algebraically localized. [S0031-9007(98)06120-1]

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A major feature of quantum dynamics of classically chaotic systems is the quantum suppression of classical chaotic excitations, a phenomenon known as dynamical localization. A prototype model, both for classical chaos and quantum dynamical localization is the kicked rotator model [1] (KRM), whose dynamics is described by the well-known Chirikov standard map [2] (CSM). This is a 2D continuous perturbed twist map, with a transition point, discriminating between bounded motion [prevalently regular on invariant Kolmogorov-Arnold-Mose (KAM) tori] and unbounded and diffusive one (prevalently chaotic). Even though transport properties of 2D maps are now quite well understood, analytical results are possible only for particular maps, e.g., linear [3]. In particular, the latter are the simplest discontinuous perturbed twist maps on the cylinder. For such discontinuous maps the hypothesis of the KAM theorem is not satisfied and the motion is typically unbounded even if it is possible to mark two different dynamical regimes (both diffusive). Discontinuous maps also emerge from the study of more concrete physical models, such as the motion of a particle colliding elastically within a two-dimensional bounded region (billiard [4]). On the other side very little is known about the quantum dynamics of such discontinuous maps. In particular, it is far from being obvious that the relation between quantum localization and classical diffusion, obtained for the KRM, holds in this case too.

To answer the above questions, let us consider the following discontinuous map on the cylinder $[0,2\pi)\times[-\infty,\infty]$:

$$\bar{p} = p + kf(\theta), \qquad \bar{\theta} = \theta + T\bar{p}, \quad \bmod - 2\pi, \tag{1}$$

where $f(\theta) = \sin(\theta)\,\mathrm{sgn}(\cos\theta)$. This function is a particularly simple approximation of the stadium map [4,5]. Moreover, it is quite similar to the CSM [where $f(\theta) = \sin\theta$] which has been widely investigated in the past.

Even if the following analysis has been put forward for this specific function, it can be generalized [5] to generic discontinuous, periodic, and bounded [$|f(\theta)| \leq 1$] functions. This set of functions can be also enlarged to continuous bounded functions with a discontinuous derivative. In this case the situation is slightly complicated, since usually a critical value of the parameter $K = kT$ appears (see [6] for the piecewise linear map) such that, when $K < K_{cr}$, the phase space is covered by invariant tori which do not permit unbounded motion along the cylinder: only for $K > K_{cr}$ the motion is diffusive.

The CSM is characterized by unbounded diffusive motion in the momentum $p$ for $kT > 1$ while, when $kT < 1$, the motion is prevalently regular with regions of stochasticity bounded by KAM invariant circles. On the other hand, the classical properties of map (1) are quite different. Indeed, due to the discontinuities of $f(\theta)$ at $\theta = \pi/2, 3/2\pi$, the hypothesis of the KAM theorem is not satisfied and, generally speaking, KAM tori do not exist, even for very small $k$. This means that one trajectory fills, in a dense way, any portion of the cylinder (phase space), for any $k \neq 0$. Nevertheless cantori can be proven to exist as for the continuous case [7]. Moreover, since KAM tori suddenly disappear for any small $k$, it is reasonable to guess that most of the phase space will be covered by cantori (remnants of KAM invariant tori) that constitute partial barriers to the motion [8]. Because of the sticking of trajectories along these invariant structures the diffusive motion is slowed down in close analogy to the saw-tooth map case described in [3].

An example of the classical map dynamics is given in Fig. 1. In the right picture [1(b)], the Poincaré surface of section is shown for the discontinuous map (1). A single initial condition has been iterated $n = 3\times10^4$ times. As one can see a single particle is free to wander in the whole phase space but the motion is far from being random. Indeed, due to sticking in the neighborhood of cantori, the trajectory is almost regular on a finite time scale $\tau$. Diffusive motion results from jumping among different stable varieties belonging to different cantori. As the iteration time, or the number of initial particles, is increased, regular structures disappear and the surface of section appears to be covered uniformly. For the sake of comparison in the left picture [1(a)] the same portion of

  

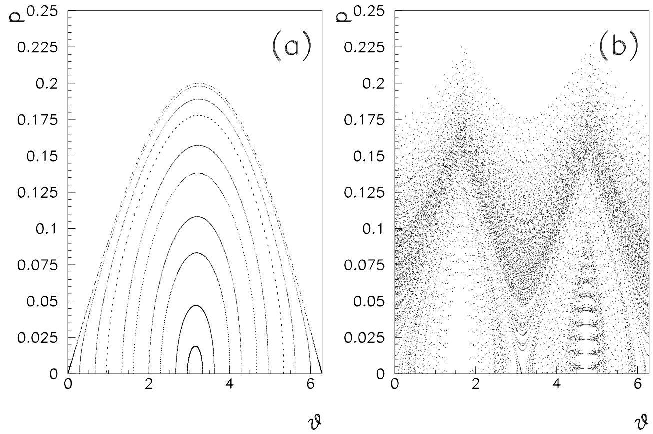


FIG. 1. Poincarè surface of section for $k = 0.01$, $T = 1$. (a) Ten different particles with initial momentum $p = 0.001$ and different phase $\theta$ have been iterated $n = 10^3$ times for the Chirikov standard map. (b) One particle starting at the point $p_0 = 0.011$ and $\theta_0 = 3$ has been iterated $3 \times 10^4$ times using the discontinuous function $f(\theta)$.

phase space is shown for the CSM, with the same value of $k$. Here ten different trajectories have been iterated $n = 10^3$ times: each trajectory covers just one torus.

Despite the "quasi" regularity of the motion, numerical results show that, when $kT < 1$, the dynamics is diffusive, after a transient time, along the cylinder axis ($p$ coordinate) with a diffusion rate $D$ given by

$$D = \lim_{n\to\infty} \frac{\langle p^2(n)\rangle}{n} = D_0 k^{5/2}\sqrt{T}\,, \qquad (2)$$

where $n$ is the time measured in iterations of the map (1) and the average $\langle\cdots\rangle$ has been performed over an initial ensemble of particles with the same momentum $p$ and random phases $0 < \theta < \pi$. Also, in (2), $D_0 \simeq 0.4$ is a numerical constant [dependent on the function $f(\theta)$] and the factor $\sqrt{T}$ has been added for dimensional reasons. The origin and the parametric dependence of the transient time that could be at the roots of the exponent $5/2$ requires further investigation.

On the other side, when $kT > 1$, the random phase approximation [2] can be applied and one finds diffusive motion along the $p$direction with a diffusion rate $D \simeq D_{ql} = k^2/2$, where $D_{ql}$ is the diffusion rate in the quasilinear approximation, namely, assuming the phases $\theta$ to be completely random uncorrelated variables. Notice that, in the undercritical region $kT < 1$, the diffusion coefficient $D \simeq k^2\sqrt{kT}$ is less than the quasilinear one $D_{ql} \sim k^2$, due to the sticking of trajectories close to cantori. In Fig. 2 the dependence of the diffusion rate $D$ is shown as a function of $k$ for $T = 1$. The dashed and full lines indicate, respectively, the quasilinear diffusion ($kT > 1$) and the slow diffusion ($kT < 1$).

The apparently strange dependence of $D$ on $k$, in the "slow" diffusive case $kT < 1$ was found in similar discontinuous maps, e.g., the saw-tooth map [3] [$f(\theta) = \theta/2\pi$], or the Stadium map [4]. In Ref. [3] a theoretical explanation of the exponent $5/2$ was given in terms of a Markovian model of transport based on the partition of phase space into resonances.

Let us now consider the quantized version of map (1). According to a well-known procedure [1] the quantum dynamics can be studied by iterating the quantum evolution operator over one period $\mathcal{U}_T$, starting from an initial state $\psi_0(\theta)$

$$\psi(T) = \mathcal{U}_T\psi_0 = e^{-i\hbar T\hat{n}^2/2} e^{-ikV(\theta)/\hbar}\psi_0\,. \qquad (3)$$

In (3), as usual, $\hat{n} = -i\hbar\partial/\partial\theta$ and $V(\theta) = |\cos\theta|$. Quantum dynamics depends on both parameters $k/\hbar$ and $T\hbar$ separately. These parameters can be renormalized by letting $k/\hbar \to k$ and $T\hbar \to T$ (which is the same as to put $\hbar = 1$). The semiclassical limit is then recovered by performing simultaneously the limits $k \to \infty$ and $T \to 0$ keeping $kT =$ const.

The most studied example of quantization of twist maps like (1) is the KRM [1], where $V(\theta) = \cos\theta$. Nevertheless the regime $kT < 1$, different from the case $kT > 1, k \gg 1$, was not the object of intense investigations.

At least numerically, one can observe two different regimes, distinguished by the so-called Shuryak border $k = T$ [9]. For $k > T$ the quantum steady state is exponentially localized over a number $l_\sigma \simeq \sqrt{k/T}$ of momentum states [10,11]. This number has been interpreted [10], in a realistic way, as the number of quantized momentum states contained in the main classical resonance [see Fig. 1(a)] the size of which is $\sqrt{k/T}$ [2]. For $k < T$ the width of the principal resonance is smaller than the distance among quantized momentum levels, and no kind

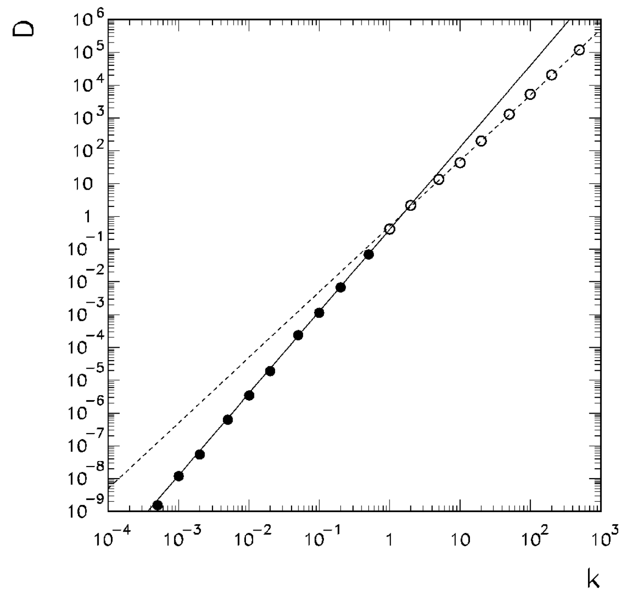


FIG. 2. Diffusion rate for the discontinuous map as a function of $k$ and $T = 1$. Open and full circles indicate, respectively, the "quasilinear" and the slow diffusion. The dashed line represents the quasilinear approximation $D = D_{ql} = 0.5k^2$ which holds for $k > 1$. The full line is the best fit $D = 0.4k^{5/2}$ obtained from full circles.

of semiclassical excitation process, based on the overlapping of resonances, is possible. In the following, the analysis will then be restricted to the case $k > T$ only.

Before studying the discontinuous case, let us recall a few important facts related to the evolution operator (3). Because of the discontinuity in the first derivative of the potential $V(\theta)$, the matrix elements of $\mathcal{U}_T$ in the momentum basis decay according to a power law away from the principal diagonal: $|\mathcal{U}_{n,n+s}| \simeq 1/s^2$. This case was investigated [12] for band random matrices: it was found to be typically characterized by power-law localized eigenstates around their centers $n_0$, $|\phi_n| \simeq |n - n_0|^{-2}$.

The following question is then important: Is it possible to connect quantum localization lengths and classical diffusion rates, as in the case of the KRM? If so, what is the critical border necessary to start the classical-like diffusion process? Moreover, what is the role played by classical invariant structures, such as cantori, in quantum dynamics?

To answer the last question, let me recall the pioneering works [13,14] where quantum propagation of wave packets through the classical cantori was first investigated. Other important results can be found in [15] where it was proposed that cantori could act, in quantum mechanics, as total barriers to the motion if the flux exchanged through turnstiles is less than $\hbar$. One can then reasonably assume that, in the deep quantum regime, the system will not be able to "see" the holes in the cantori that behave as classical invariant tori.

A more refined analysis requires the introduction of some kind of measure of the quantum distribution width. Since in this model localization is presumably not exponential, a unique scale of localization is not properly defined. For instance, while in the case of exponential localization the usual measures of localization, e.g., inverse participation ratio, variance, entropy [11], coincide, for power-law localized distributions the dependence on the parameters can be different if different definitions are adopted. Then we choose the variance as a measure of the distribution extension (degree of localization):

$$l_\sigma = \left[ \sum_n n^2 |\psi_n(t)|^2 \right]^{1/2}, \tag{4}$$

which has a proper semiclassical limit. Since this is, in general, an oscillatory function of the iteration time, a further average in time is necessary in order to get time-independent results.

Numerical data are presented in Fig. 3 where $l_\sigma$ has been plotted as a function of $\sqrt{k/T}$. Excluding oscillations, data follow, for $k < k_{cr}$, the dotted line $\sqrt{k/T}$, as for KRM. Indeed, as one can see comparing Figs. 1(a) and 1(b) the principal resonance and the quasiprincipal resonance have roughly the same size. This is a manifestation of the regularity imposed by quantum mechanics, or, in other words, of the discrete nature of the quantum phase space. This means that classical discontinuous structures behave exactly as continuous ones.

On the other side, since the classical discontinuous system is diffusive, the number of occupied quantum states should increase on going into the semiclassical region. Following known arguments for the dynamical localization, one can expect the localization length to be given by the number of states inside a quasiprincipal resonance ($\sqrt{k/T}$), as soon as it equals numerically the classical diffusion coefficient. In this way the critical value $k_{cr}$ can be obtained by equating the following expressions:

$$l_\sigma \simeq \sqrt{k/T} \simeq D = D_0 k^{5/2} \sqrt{T} \tag{5}$$

that gives the value $k_{cr} = 1/\sqrt{D_0 T}$.

It is important to notice that the "quasi-integrable" value $l_\sigma \simeq \sqrt{k/T}$ can survive well above the threshold $k = 1$ that is the value necessary to start the classical-like diffusion process for the KRM when $kT > 1$. Also, this kind of localization is not connected with any classical-like diffusive process, resulting instead from a quasiperiodic motion. The absence of diffusive quantum motion, in the region $T < k < k_{cr}$ can be ascribed to a "dynamical" diffusion rate $l_\sigma$ less than the size of the quasiprincipal resonance $l_\sigma \simeq \sqrt{k/T}$. For instance, numerical simulation indicates a localization length $l_\sigma \simeq 80 \pm 10 \simeq \sqrt{k/T}$ for $k = 10 \gg 1$, $T = 1/1000$, while $D \simeq 0.4 k^{5/2} \sqrt{T} = 4$. In other words, in the region dominated by slow diffusion, the threshold for classical-like diffusion is $k > k_{cr} = 1/\sqrt{D_0 T}$ and not $k > 1$.

These theoretical predictions are confirmed by the numerical data presented in Fig. 3, which closely follow

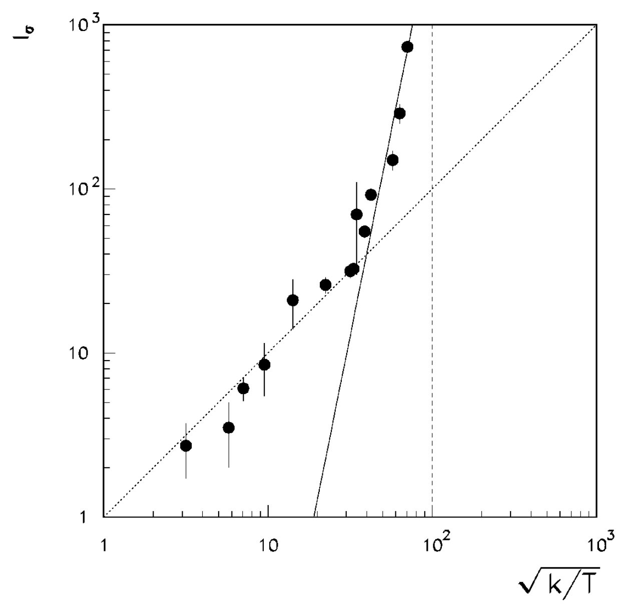


FIG. 3. Localization length $l_\sigma$ as a function of $\sqrt{k/T}$ for fixed $T = 0.01$. Lines are the theoretical predictions: dotted line ($l_\sigma = \sqrt{k/T}$); full line ($l_\sigma = D$); dashed line is the quasilinear border $k_{ql} T = 1$.

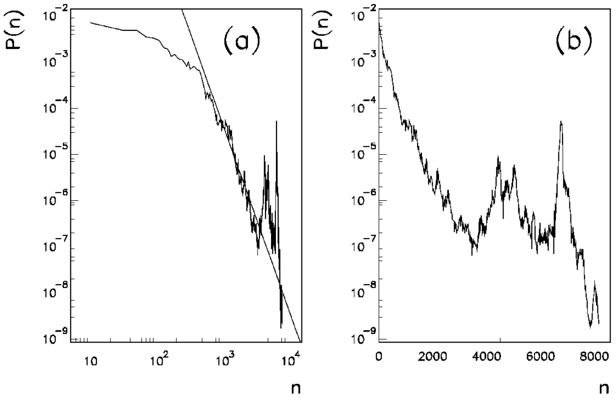


FIG. 4. Quantum stationary distribution for $k = 50$, and $T = 0.01$. The map has been iterated $10^6$ times. The final distribution is obtained by averaging over the last $10^5$ kicks. The initial state is $\psi_n = \delta_{n,0}$. (a) Log-log scale. The line $n^{-4}$ has been drawn to guide the eye. (b) Log scale.

the curve (full line) $l_\sigma = D$ for $k_{cr} < k < k_{ql}$. Here $k_{ql}$ stands for the border of validity of quasilinear diffusion: $k_{ql} = 1/T$. This confirms and extends the validity of the dynamical localization theory even in the presence of slow diffusion and algebraic decay. This last point can be directly observed in Fig. 4(a) where the quantum steady state distribution $P(n)$ is shown together with the corresponding line $n^{-4}$.

The dynamical localization mechanism is not connected with this power-law decay. Indeed the same algebraic decay can be found for any $k$, in the region $k < k_{cr}$ as well for $k > k_{ql}$ (at least in the tails of the distribution [5]). In more details, on semiclassically approaching the border $k_{ql}$, the quantum distribution shows big peaks of probability for high momentum values that indicate that new regions of the classical phase space are now quantally accessible. It is exactly the presence of such peaks that causes the large increase of $l_\sigma$. The presence of bumps of probability far from the initial state $n_0 = 0$ is shown in Fig. 4(b).

In conclusion, a discontinuous map which is a simple generalization of the Chirikov standard map has been studied. Differently from the latter, the dynamics is slowly diffusive even when the motion described by the CSM is prevalently regular. In this region the quantum analysis reveals quite unexpected features. Above the Shuryak border $k > T$, two different scaling laws for the localization length are found. The first, $l_\sigma \simeq \sqrt{k/T}$, marked by the presence of classical cantori acting as total barriers to quantum motion, is a region of quantum integrability. The second is a region characterized by dynamical localization ($l_\sigma \simeq D$) thus indicating the existence of this phenomenon even in the case of slow diffusion. At the critical point $k_{cr}$, separating these regimes, quantum dynamics starts to follow the classical excitation process. Differently from the KRM, for which $k_{cr} \simeq 1$, one finds here $k_{cr} \simeq 1/\sqrt{T}$.

During the completion of this Letter I became aware of another related work [16] where a regime of quantum integrability is found, for the Stadium billiard, in the region delimited by the inequalities $E\epsilon > 1$ and $\sqrt{E}\,\epsilon^2 < 1$, where $E$ is the energy of the particle and $\epsilon \ll 1$ is the ratio between the straight line and the circle radius [4]. The billiard dynamics is well described [5] in terms of the map (1) via the substitutions $k = 2\epsilon\sqrt{E}$, $T = \sqrt{2/E}$. It is then easy to verify that the quantum-integrable regime found in Ref. [16] $E^{-1} < \epsilon < E^{-1/4}$ coincides with the regime dominated by classical cantori $T < k < 1/\sqrt{T}$.

This may be a first indication that not only the classical, but also the quantum dynamics of the Stadium, can be described in terms of maps: This will be the subject of a future work [5].

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