Quantum Hamilton-Jacobi Equation

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The nontrivial transformation of the phase space path integral measure under certain discretized analogs of canonical transformations is computed. This Jacobian is used to derive a quantum analog of the Hamilton-Jacobi equation for the generating function of a canonical transformation that maps any quantum system to a system with a vanishing Hamiltonian. A perturbative solution of the quantum Hamilton-Jacobi equation is given. This solution gives a new way to compute quantum corrections for any soliton equation for which action-angle variables are known. [S0031-9007(98)06163-8]

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A remarkable formulation of classical dynamics is provided by the Hamilton-Jacobi equation: If $S(q,P,t)$ satisfies

$$\frac{\partial S}{\partial t}(q,P,t) + H(q,\partial_q S,t) = 0\,, \tag{1}$$

where H is the Hamiltonian, then the canonical transformation defined by

$$\partial_P S = Q, \qquad \partial_q S = p \tag{2}$$

maps the dynamical system governed by the Hamiltonian H to a trivial dynamical system, one with vanishing Hamiltonian. To see this, note that $p\dot{q} - H = \partial_q S\dot{q} - H = \frac{d}{dt}(S - PQ) + P\dot{Q}$, using Eq. (1). Boundary terms do not affect the phase space equations of motion, so this mapping determines identical classical dynamics [1]. The function S is Hamilton's principal function, or action, which acquires a greater significance in quantum mechanics [2,3].

Quantum mechanically, canonical transformations of the form considered above do not generate equivalent quantum systems [4–8]. There is no natural action of the group of symplectic transformations on the quantum Hilbert space. Alternatively, in Feynman's formulation of quantum mechanics [3], the phase space path integral is not invariant under canonical transformations. The noninvariance of phase space (and coordinate space) path integral measures has been the focus of a great deal of work [6–8]. In the present work, the general problem of symplectic transformations will *not* be considered—I shall just consider the properties of the phase space path integral under the discretized analogs of canonical transformations of a particular type. (The approach taken in this Letter is closest to that of Ref. [8].) The motivation is to answer the following: Is there a deformation of Eq. (1) which allows a quantum mechanical map from an arbitrary quantum system to one with a vanishing Hamiltonian? Apart from the fundamental interest in this question, the main application is to the quantization of solitons, of especial interest since the quantum properties of solitons are at the heart of recent developments in string theory [9]. Equation (14) gives a simple method of computing quantum corrections to classical solutions, when classical action-angle variables are known.

After a short review of the path integral formulation to make the measure precise, I will compute the transformation of the measure under the transformations that keep the discretized $\int p dq$ term in the action invariant (up to total derivatives). These transformations differ from canonical transformations due to the discretization of the phase space path integral, so the Jacobian for the change of variables in the path integral is nontrivial. A particular application of this result gives the desired deformation of the Hamilton-Jacobi equation, with deformation parameter the Planck constant. From this, the quantum Hamilton-Jacobi equation, Eq. (14), is immediate. The solution of Eq. (14) as a formal perturbative series takes a simple form, Eq. (15).

We compute $\langle q'', t''|p', t'\rangle$ as a functional integral, choosing the momentum state to position state amplitude to obtain a symplectically invariant form for the path integral measure. Note $\langle p|q\rangle = (2\pi)^{-d/2}\exp(-ipq)$, and if H is ordered so that all momentum operators appear on the left, $\langle p|H|q\rangle = (2\pi)^{-d/2}\exp(-ipq)H(q,p)$. Assume that the Hamiltonian is time independent for notational simplicity, since the generalization to arbitrary Hamiltonians is trivial. Since $\langle q'', t''|p', t'\rangle = \lim_{N\uparrow\infty}\langle q''|(1 - i\epsilon H)^N \times |p'\rangle$, with $\epsilon \equiv (t'' - t')/N$, using $1 = \int dp\, dq |p\rangle \times \langle q|(2\pi)^{-d/2}\exp(-ipq)$ between every factor of $(1 - i\epsilon H)$, we find

$$\langle q'', t''|p', t'\rangle = \frac{1}{\sqrt{2\pi}}\lim_{N\uparrow\infty}\int\prod_{i=1}^{N}\frac{dp_i dq_i}{(2\pi)^d}\, e^{iA_N}e^{ip_0 q_1}, \tag{3}$$

where $A_N \equiv \sum_{i=1}^{N}[p_i(q_{i+1} - q_i) - \epsilon H(p_i,q_i)]$. Here, $q_{N+1} \equiv q''$ and $p_0 = p'$, and q_1 and p_N are integrated over. In the continuum limit, $A_N \to A_\infty \equiv \int dt[p\dot{q} - H]$, and the measure can be described heuristically as an integration over all phase space paths satisfying $q(t'') = q''$, $p(t') = p$, with $p(t'')$ and $q(t')$ integrated over. For the pitfalls in such continuum descriptions, see [4–8].

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Equation (3) can now be used to consider the properties of the phase space path integral under canonical transformations. The measure $\prod dp_i dq_i$ is clearly invariant under arbitrary i-dependent canonical transformations as a straightforward mathematical fact. However, A_N is not invariant under such transformations. The point of the following exercise is to find a transformation of integration variables $(p_i, q_i) \longrightarrow (P_i, Q_i)$ that changes the pdq term in A_N in a simple way, and then to compute the Jacobian for this transformation.

Consider defining functions $Q(q,p), P(q,p)$ implicitly by means of the following definitions, for arbitrary functions $S_i(P,q)$:

$$\begin{aligned} p_i(q_{i+1} - q_i) &\equiv S_i(q_{i+1}, P_i) - S_i(q_i, P_i), \\ Q_i(P_i - P_{i-1}) &\equiv S_{i-1}(q_i, P_i) - S_{i-1}(q_i, P_{i-1}). \end{aligned} \tag{4}$$

Now observe that

$$p_i(q_{i+1} - q_i) + Q_i(P_i - P_{i-1}) = [S_i(q_{i+1}, P_i) - S_{i-1}(q_i, P_{i-1})] - [S_i(q_i, P_i) - S_{i-1}(q_i, P_i)], \tag{5}$$

with the first term in $[\cdots]$ a telescoping series when summed over i. Note that Eq. (5) has no dependence on H. Thus one finds

$$A_N = \sum_{i=1}^{N} [-Q_i(P_i - P_{i-1}) - \epsilon H(p_i, q_i) - \{S_i(q_i, P_i) - S_{i-1}(q_i, P_i)\}] + \text{boundary terms}. \tag{6}$$

Comparing Eq. (6) with Eq. (1), this is the form expected if time is discretized. I must now compute the effect of the substitutions in Eq. (4) on the measure.

Keeping P_{i-1}, q_{i+1} fixed, I find that

$$dp_i dq_i = (q_{i+1} - q_i)^{-1} \partial_{P_i}[S_i(q_{i+1}, P_i) - S_i(q_i, P_i)] dP_i dq_i, \tag{7}$$

whereas

$$dP_i dQ_i = (P_i - P_{i-1})^{-1} \partial_{q_i}[S_{i-1}(q_i, P_i) - S_{i-1}(q_i, P_{i-1})] dP_i dq_i. \tag{8}$$

The Jacobian for the change of variables $(p,q)_i \longrightarrow (P,Q)_i$ is therefore nontrivial. It is not possible to proceed further without some knowledge of the relation between the canonical variables with subscripts i and the variables with subscripts $i \pm 1$, in other words, without some restriction on the sequences q_i and P_i as $N \uparrow \infty$. I will come back to these restrictions momentarily.

At a formal level, *assuming that $P_{i-1} - P_i$ and $q_{i+1} - q_i$ are small as $N \uparrow \infty$*, it follows from Eqs. (7) and (8) that

$$\begin{aligned} dp_i dq_i &= \left[\partial_{P_i}\partial_{q_i} S_i(q_i, P_i) + \frac{1}{2}(q_{i+1} - q_i)\partial_{P_i}\partial_{q_i}^2 S_i(q_i, P_i) + \ldots\right] dP_i dq_i, \\ dP_i dQ_i &= \left[\partial_{P_i}\partial_{q_i} S_{i-1}(q_i, P_i) - \frac{1}{2}(P_i - P_{i-1})\partial_{P_i}^2\partial_{q_i} S_{i-1}(q_i, P_i) + \ldots\right] dP_i dq_i. \end{aligned} \tag{9}$$

We can also derive the analog of Eq. (9) for $dq_{i+1} dp_i$:

$$\begin{aligned} dq_{i+1} dp_i &= \left[\partial_{P_i}\partial_{q_{i+1}} S_i(q_{i+1}, P_i) - \frac{1}{2}(q_{i+1} - q_i)\partial_{P_i}\partial_{q_{i+1}}^2 S_i(q_{i+1}, P_i) + \ldots\right] dq_{i+1} dP_i, \\ dQ_{i+1} dP_i &= \left[\partial_{P_i}\partial_{q_{i+1}} S_i(q_{i+1}, P_i) + \frac{1}{2}(P_{i+1} - P_i)\partial_{P_i}^2\partial_{q_{i+1}} S_i(q_{i+1}, P_i) + \ldots\right] dq_{i+1} dP_i. \end{aligned} \tag{10}$$

Equations (9) and (10) determine Jacobians that differ by the sign of the total time derivative contribution, indicating that this is a nonuniversal artifact of the discretization. Such contributions are, of course, to be expected, since the relation of the index i to the continuum time variable t for q, P, and S need not be the same. We use the ultralocality of the phase space measure to eliminate this total derivative contribution by averaging the Jacobians determined by Eqs. (9) and (10)—heuristically, one can interpret this as setting the time associated with P_i midway between q_i and q_{i+1}. Note that *all* powers of $q_{i+1} - q_i, P_i - P_{i-1}$ cancel in this average, *before* taking the continuum limit. Therefore, anomalous contributions of the Edwards-Gulyaev [6] type do not appear. So, finally, assuming that S_i is chosen to become a differentiable function of t as $N \uparrow \infty$, we find

$$\lim_{N\uparrow\infty} \prod dp_i dq_i = \lim_{N\uparrow\infty} \prod dP_i dQ_i \times \exp\left[\frac{1}{2}\int dt\, \partial_t \ln\det \partial_P \partial_q S(q, P, t)\right]. \tag{11}$$

Equation (11) has *exactly* the form that one expects, *in the continuum limit*, since successive canonical transformations obey a group law that is consistent with the $\ln\det \partial_P \partial_q S$ form of the Jacobian. This is an important consistency check on the calculation.

We can check this Jacobian by performing an explicit calculation in any quantum mechanics problem, since the measure's transformation properties are universal, i.e., independent of the Hamiltonian. A simple choice of Hamiltonian is $H = \frac{1}{2}(p^2 + q^2)$, the harmonic oscillator. In this case, one knows [3] that

$$\langle q'', t'' | p', t' \rangle = \frac{1}{\sqrt{2\pi \cos(t - t')}} \times \exp\left[-\frac{i}{2} \tan(t - t')[(p'^2 + q''^2) - 2p'q'' \csc(t - t')]\right]. \tag{12}$$

Choose $S(q, P, t) \equiv qP \sec(t - t') - (q^2 + P^2)\tan(t - t')/2$. This choice of S amounts to $P = p\cos(t - t') + q\sin(t - t')$, with $Q = q\cos(t - t') - p\sin(t - t')$, and satisfies the classical Hamilton-Jacobi equation, Eq. (1). According to the calculations above [Eqs. (3) and (6)], performing some trivial integrations the transition amplitude should equal

$$\langle q'', t'' | p', t' \rangle = \frac{1}{\sqrt{2\pi}} \int \frac{dP_N dQ_1}{2\pi} \times e^{iS(t'')} e^{iQ_1(p' - P_N)} e^{\frac{1}{2}\ln\sec(t - t')}. \tag{13}$$

Comparing this form to Eq. (12), we find exact agreement. This is another check on the absence of Edwards-Gulyaev [6] corrections, since the harmonic oscillator is not a cyclic Hamiltonian in p, q coordinates.

Equations (4) and (11) imply $\int dt[p\dot{q} - H(p,q)] \rightarrow \int dt[p(P,Q)\dot{q}(P,Q) - H(p(P,Q), q(P,Q)) - \frac{i}{2}\partial_t \times \ln\det \partial_P \partial_q S]$. Thus, using Eq. (6) and restoring $\hbar$, if S satisfies

$$\partial_t\left(S + \frac{i}{2}\hbar \ln\det \partial_P \partial_q S\right) + H(q, \partial_q S) = 0, \tag{14}$$

Eq. (4) will map the quantum system to a quantum system with a vanishing Hamiltonian. The telescoping terms in Eq. (5) give rise to boundary terms in the path integral of $\exp[iS(P(t''), q(t''), t'')]$ and $\exp[-iS(P(t'), q(t'), t') + ip(t')q(t')]$.

What are the conditions for the validity of the formal manipulations that lead from Eqs. (7) and (8) to Eq. (11)? The measure on phase space with the Hamiltonian H must be concentrated on paths such that $q_{i+1} - q_i$ tends to zero with ϵ, and similarly for $P_i - P_{i-1}$ with the measure determined by the transformed Hamiltonian. This is true with quite mild restrictions [5] on $H(p,q)$ for q, and similar restrictions on $H'(P,Q) \equiv H(p(P,Q), q(P,Q)) + \partial_t[S + i/2 \ln\det \partial_P \partial_q S]$ for P. The smoothness of P paths is trivially true after the change of variables if S satisfies Eq. (14), since the action is just $-\int dt Q\dot{P}$. In this context, it should be noted that the form of the transformed Hamiltonian, H', is valid *only* in the $\epsilon \downarrow 0$ limit—however, since the Jacobian is explicitly ultralocal, several types of anomalous contributions that appear for general symplectic transformations [6–8] *do not appear for the specific symplectic transformations considered* in this Letter. The applications of Eq. (14) to field-theoretic problems may be more interesting, for ordering difficulties in field theory are usually absorbed into renormalization constants [5].

Equation (14) may appear to be a simple deformation of Eq. (1), but in fact it is not. According to Jacobi's theorem [1], finding a sufficient number of solutions of Eq. (1) allows one to solve the dynamics of the system—the key point is that the variables P are integration constants for these solutions, an interpretation possible since they do not appear in Eq. (1) explicitly. This interpretation is not possible for Eq. (14), so *a priori* one has to find appropriate choices of P before one can even attempt to solve this equation, unless one treats $\hbar$ as a perturbation parameter. Since such a perturbative solution is not a good approximation in general, one may be led to conclude that Eq. (14) is of less practical value in quantum mechanics than Eq. (1) is in classical mechanics. Nevertheless, Eq. (14) is simple, and of conceptual value in understanding the classical limit of quantum mechanics. A formal solution to Eq. (14) can be found as follows: Let $S \equiv S_0 + \hbar S_1 + \hbar^2 S_2 + \ldots$. Then

$$\begin{aligned}
&\partial_t S_0(q,P,t) + H(q, p = \partial_q S_0, t) = 0, \\
&\partial_t S_1(q,P,t) + \partial_p H(q, p = \partial_q S_0, t)\partial_q S_1(q,P,t) = -\frac{i}{2}\operatorname{tr}[(\partial_P \partial_q S_0)^{-1}\partial_t \partial_P \partial_q S_0], \\
&\partial_t S_2(q,P,t) + \partial_p H(q, p = \partial_q S_0, t)\partial_q S_2(q,P,t) = -\frac{i}{2}\partial_t \operatorname{tr}[(\partial_P \partial_q S_0)^{-1}\partial_P \partial_q S_1], \\
&\partial_t S_3(q,P,t) + \partial_p H(q, p = \partial_q S_0, t)\partial_q S_3(q,P,t) = -\frac{i}{2}\partial_t \operatorname{tr}\{(\partial_P \partial_q S_0)^{-1}\partial_P \partial_q S_2 - \tfrac{1}{2}[(\partial_P \partial_q S_0)^{-1}\partial_P \partial_q S_1]^2\} \ldots
\end{aligned} \tag{15}$$

The solution to this set of equations is obtained by the method of characteristic projections. Let S_0 be a complete integral of Eq. (1), which of course coincides with the first equation in Eq. (15), and $q(t)$ a solution of $\dot{q} = \partial_p H(q(t), p = \partial_q S_0, t)$, which is just one of the classical equations of motion. Then $S_1(q(t), P, t)$ is a solution of

$$\frac{d}{dt} S_1 = -\frac{i}{2} \operatorname{tr}[(\partial_P \partial_q S_0)^{-1} \partial_t \partial_P \partial_q S_0](q(t), P, t), \tag{16}$$

with analogous equations for S_i, $i > 1$. We see, therefore, that the integral surfaces, indexed by P, of Eq. (15), depend on the behavior of integral surfaces *as functions of* P. Thus, the perturbative solution of Eq. (14) incorporates information about quantum fluctuations by its dependence on the complete integral of Eq. (1) at neighboring values of P.

Equation (11) shows that the transformation to classical action-angle variables leaves behind a nontrivial Hamiltonian, $\frac{i}{2}\hbar\partial_t \ln\det \partial_P \partial_q S$, which takes into account quantum fluctuations. Classical canonical transformations that solve Eq. (1), and satisfy $\partial_q \det \partial_P \partial_q S = \partial_P \det \partial_P \partial_q S = 0$, will also solve the quantum dynamics, with the anomalous term serving as a computation of the fluctuation determinant about classical solutions, as in the harmonic oscillator considered above. This is of interest for quantizing solitons, in situations where action-angle variables are known for the classical field equation. Equation (15) gives then an explicit and simple way to compute quantum corrections to soliton dynamics.

The formulation considered above for canonical transformations may be too limited. The variables P have a fundamentally different role to play in Eq. (14) as compared to Eq. (1), and it may be natural to look for solutions in which P, Q describe a noncommutative symplectic manifold. This is suggested by the fact that the quantum energy spectrum could have discrete and/or continuous components, and such a space cannot always be described as a commuting symplectic manifold [10]. In such a case the form of the anomaly will be different. It would be fascinating if quantum mechanics on a commuting phase space could be mapped to a vanishing Hamiltonian on a (possibly) noncommuting phase space.

To conclude, I mention that two recent works [11,12] have addressed related issues. In [11], it is claimed that the complete solution of the classical Hamilton-Jacobi equation, Eq. (1), determines the quantum mechanical amplitude by means of a single momentum integration instead of a path integral. While the path integration of the trivial quantum mechanics with vanishing Hamiltonian indeed reduces to (a variant of) a phase space integration as mentioned above [and explicitly found in the case of the harmonic oscillator, Eq. (13)], Eq. (14) is distinct from the classical equation, so it appears to contradict [11]. Reference [12] postulates a diffeomorphic covariance principle, based partly on an $SL(2, C)$ algebraic symmetry of a Legendre transform, and finds a modification of the classical Hamilton-Jacobi equation that has appropriate covariance properties for the postulated equivalence. Their function S satisfies an equation quite different from Eq. (14), and it is argued that S is related to solutions of the Schrödinger equation. Functional integrals of any sort do not appear in [12], and there is no relation to the present result, Eq. (11).

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