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Decoherence, Chaos, and the Correspondence Principle

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We present evidence that decoherence can produce a smooth quantum-to-classical transition in nonlinear dynamical systems. High-resolution tracking of quantum and classical evolutions reveals differences in expectation values of corresponding observables. Solutions of master equations demonstrate that decoherence destroys quantum interference in Wigner distributions and washes out fine structure in classical distributions, bringing the two closer together. Correspondence between quantum and classical expectation values is also reestablished. [S0031-9007(98)06134-1]

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The status of quantum-classical correspondence for dynamical systems is somewhat unclear and perhaps even controversial. In a nonlinear system, a single classical trajectory initially centered on a quantum wave packet can quickly diverge from the motion of the centroid given by the quantum expectation values of x and p (this defines the Ehrenfest time, cf. Ballentine *et al.* in Ref. [1]). However, when the single trajectory is replaced by a classical probability distribution to give the correspondence principle a better chance, the situation is far less certain. Thus, some authors have reported evidence for a breakdown of the quantum-classical correspondence in chaotic systems [2], while others have argued that it can be preserved when stated in terms of the expectation values of dynamical variables [1]. In yet another line of investigation it has been shown that, even in chaotic systems, semiclassical methods are successful for times longer than was previously believed possible [3].

In integrable systems, a rapid divergence between classical and quantum evolutions can occur for initial conditions near an unstable point: This can force the system to undergo a "double-slit experiment," yielding very different outcomes in the two cases. This breakdown of correspondence is caused by the coherent interference of fragments of the wave packet and occurs on a short time scale, on the order of the system dynamical time. However, for generic initial conditions, quantum and classical expectation values are expected to diverge on a time scale inversely proportional to some power of $\hbar$ [4,5]. This is regarded as sufficiently slow to cause no difficulties with the classical limit of quantum theory.

The first purpose of this Letter is to investigate the quantum-classical correspondence at the level of expectation values for chaotic dynamical systems, where a breakdown may be anticipated on a much smaller *logarithmic* time scale $t_\hbar \sim \ln(C/\hbar)$ [4–6]. We show below that such a loss of correspondence *does* occur, though its magnitude is not in conflict with earlier results [1]. While the time scale on which the violation occurs varies with the particular initial condition chosen, it is consistent with the logarithmic time $t_\hbar$.

In order to effect a quantum-classical comparison, we chose Gaussian packets as initial states, with positive definite Wigner distribution functions. The time evolution was then performed using either classical or quantum dynamical equations. In the chaotic systems investigated here, for generic initial conditions (i.e., Gaussian packets randomly sampling the chaotic part of the phase space), differences between quantum and classical expectation values stay small for some time and then abruptly increase. After this divergence time, the differences remain modest, typically $\sim$5%–10% (for $\hbar = 0.1$). The Wigner function begins to differ considerably from the classical phase space distribution at relatively early times [7].

Our second purpose is to show that the discrepancy between quantum and classical evolutions is drastically decreased by even a small coupling to the environment, which in the quantum case leads to decoherence [8]. There are two limiting situations in the study of decoherence in dynamical systems. In the first case, special initial conditions (such as Schrödinger cat states) are used to study the destruction of interference already present in the initial state but with simple system dynamics, typically taken to be linear [9]. However, since quantum interference is dynamically generated only in nonlinear systems [10], the competition between generation and destruction of quantum coherence cannot be investigated. The results reported in this Letter are from a study of the second type, where the system dynamics is nonlinear, but the choice of initial states is kept deliberately simple so as to focus only on the role of dynamically induced interference (as distinct from that present in the initial state). We show below that differences at the level of expectation values are sharply reduced due to decoherence and the effect on correspondence in phase space is even more spectacular. In the parameter regime investigated here—essentially the border between quantum and classical—some remnants of quantum coherence may still survive. However, such small-scale coherence has apparently little effect on the correspondence of the expectation values.

A mechanism responsible for the quantum to classical transition should explain not just how expectation values can converge to the same answer, but also lead to compatible effective phase space distributions. A common approach is an appeal to coarse graining, a formal procedure implemented typically by convolving the individual distributions with a Gaussian distribution and then comparing the two resulting coarse-grained distributions [7]. This approach has three defects. First, as a formal mathematical procedure it can always be inverted, and thus offers no physical insight. Second, this coarse graining does not alter the dynamics, and hence cannot improve the convergence of expectation values. Third, for the classical system, the notion of a trajectory is lost and, along with it, the notion of a Lyapunov exponent.

In contrast to the coarse-graining approach, decoherence provides a *dynamical* explanation [8] of the quantum-to-classical transition by taking into account interactions with an (external or internal) environment of the system—degrees of freedom that effectively monitor and, therefore, select certain stable or "pointer" observables destined to become the classical variables [8,11]. The simplest models of this type lead to master equations for the reduced density matrix for the system [12]. Diffusion terms in these equations automatically coarse grain the distributions, now a physical effect of the coupling to the environment rather than a mathematical trick. The degree of coarse graining is determined by the interplay between the dynamics of the system and the nature and strength of the coupling with the environment. Moreover, the effectively classical master equations that describe the postdecoherence dynamics admit a Langevin description of trajectories allowing for the existence of a Lyapunov exponent. We demonstrate below that decoherence dramatically improves the correspondence of the expectation values, leads to the existence of a single phase space distribution, and allows for a Lyapunov exponent to exist at late times. All the deficiencies of the coarse-graining approach are therefore overcome.

We restrict attention to bounded, one-dimensional, driven systems. Tools employed are very high-resolution simulations of the time-dependent Schrödinger, quantum and classical Liouville, and master equations recently implemented on massively parallel computers [13]. The numerical results discussed below are for the driven system considered in Ref. [14], with Hamiltonian,

$$H = p^2/2m + Bx^4 - Ax^2 + \Lambda x \cos(\omega t). \quad (1)$$

We used a parameter regime ($m = 1$, $B = 0.5$, $A = 10$, $\Lambda = 10$, $\omega = 6.07$) in which a substantial area of phase space is predominantly stochastic, with the finite-time Lyapunov exponent $\lambda \simeq 0.4-0.5$. Gaussian phase space distributions, typically minimum uncertainty wave packets, were employed as initial conditions to sample the evolution in the stochastic region. Other Hamiltonians studied include the driven Duffing system and a two-dimensional, truncated Toda potential. These systems yielded similar results; a detailed presentation will be given elsewhere [15].

The specific model of decoherence used here is the weak coupling, high temperature limit of quantum Brownian motion [12]. In this limit, dissipation can be ignored at early times, and only the diffusive contributions in the master equations [6,8] need be kept. The diffusion constant was chosen to be small enough so that, over the time scales of interest, the energy increase was negligible, and changes in the classical phase space structure were only perturbative. For the Hamiltonian (1), the quantum master equation in terms of the Wigner function is

$$\frac{\partial f_W}{\partial t} = -\frac{p}{m}\frac{\partial f_W}{\partial x} + \frac{\partial V}{\partial x}\frac{\partial f_W}{\partial p} + L_q f_W + D\frac{\partial^2 f_W}{\partial p^2}, \quad (2)$$

where $\partial V/\partial x = 4Bx^3 - 2Ax + \Lambda\cos(\omega t)$ and L_q is

$$L_q \equiv \sum_{n\geq 1} \frac{\hbar^{2n}(-1)^n}{2^{2n}(2n+1)!}\frac{\partial^{2n+1}V}{\partial x^{2n+1}}\frac{\partial^{2n+1}}{\partial p^{2n+1}} = -\hbar^2 Bx\frac{\partial^3}{\partial p^3}.$$

The classical ($L_q = 0$) Fokker-Planck equation is

$$\frac{\partial f_c}{\partial t} = -\frac{p}{m}\frac{\partial f_c}{\partial x} + \frac{\partial V}{\partial x}\frac{\partial f_c}{\partial p} + D\frac{\partial^2 f_c}{\partial p^2}. \quad (3)$$

This dynamics is equivalent to the Langevin equation $m\ddot{x} = -V'(x) + F(t)$ where $F(t)$ denotes a Gaussian, white noise. The quantum Schrödinger and master equations, as well as the classical Fokker-Planck equations were solved using a high-resolution spectral algorithm [13]. In the absence of diffusion, the classical Liouville equation was solved as an N-body problem, the distribution being sampled by at least $\sim 10^5$ particles. Numerical checks included carrying out simulations at different spatial and temporal resolutions and a direct verification for the moments obtained from the codes by substituting them in the Bogoliubov-Born-Green-Kirkwood-Yvon-like (BBGKY-like) moment evolution hierarchy and verifying that this set of equations is satisfied to a relatively high order. Lyapunov exponents were computed using the techniques described in Ref. [16]. It was verified that, at the noise levels used, the finite-time Lyapunov exponents from the Langevin equation agreed with those computed from the Hamiltonian dynamics.

The exponential instability characteristic of chaos forces the system to rapidly explore large areas of phase space and to interfere on a time scale set by when the wave function has spread over much of the available space [4–6] and the Moyal corrections arising from its nonlocality have become comparable to the classical force [6]:

$$t_\hbar \sim \lambda^{-1} \ln(\chi \delta p/\hbar), \tag{4}$$

where λ is the Lyapunov exponent, δp is the measure of dispersion in the initial conditions, and $\chi \simeq \sqrt{|\langle \partial_x V/\partial_{xxx} V\rangle|}$ is a measure of the nonlinearity in the potential averaged over the accessible space. We investigated the evolution of several expectation values, such as $\langle x\rangle$, higher order moments such as $\langle (x - \langle x\rangle)^n\rangle$ (with a maximum $n = 4$), and expectation values of variables that, in principle, include all moments. Some of our results are shown in Fig. 1. In all cases, and for all of the investigated initial conditions, we found good agreement between the quantum and classical results during the initial portion of the evolution. (This initial period was *longer* than the Ehrenfest time, consistent with the results of Ballentine *et al.* [1]). The onset of the discrepancy depended on the initial condition, but in all cases was a factor of a few larger than the dynamical time. The discrepancy saturates to typically no more than 10% of the expectation values thereafter. The value of $\hbar$ was varied to test for logarithmic scaling: While the results are consistent with (4), the dynamic range of the simulations is insufficient to make a more precise statement.

We have therefore good evidence that, in isolated chaotic systems, the quantum-classical correspondence defined at the level of expectation values is lost relatively quickly due to dynamically generated quantum interference. This is best appreciated by comparing classical phase space densities with quantum Wigner distributions. As the wave packet spreads and folds, the Wigner function becomes dominated by small-scale interference which saturates on a scale set by the size of the system in both momentum and position: $\delta p = \hbar/L$, $\delta x = \hbar/P$. Eventually, the Wigner function is unable to track even

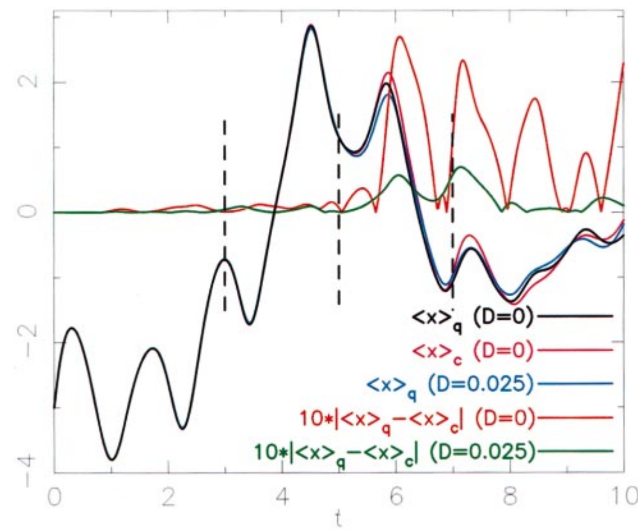

FIG. 1(color). Classical and quantum expectation values $\langle x\rangle$ as a function of time. The initial condition is a minimum uncertainty Gaussian wave packet with $\langle x\rangle = -3$, $\langle p\rangle = 8$, $\langle (x - \langle x\rangle)^2\rangle = 0.0025$, $\langle (p - \langle p\rangle)^2\rangle = 1$. $\hbar$ is set to 0.1. This yields $t_\hbar \sim 4$. The central vertical bar denotes the average divergence time, and the left and right bars denote the minimum and maximum, respectively, for ten initial conditions that randomly sampled the chaotic phase space.

the "backbone" of the classical phase space distribution and becomes a complicated looking interference pattern in which the classical phase space structure can no longer be distinguished [Fig. 2(a)]. Fine-scale structure in the interference pattern (i.e., oscillations within a $\hbar$ box) is clearly apparent. On the time scales probed in our simulations, the Wigner function has not reached the stage of "structure saturation," i.e., smoothness on a scale $\sim\hbar$ in phase space [5].

Our results are encapsulated in Fig. 2. As can be seen from comparing the decohered Wigner function [Fig. 2(b)] with the classical distribution given by the solution to Eq. (3) [Fig. 2(c)], decoherence markedly improves the correspondence at the level of distribution functions, radically changing the unitarily evolved Wigner function of Fig. 2(a) by effectively smoothing it over scales [6]; $\Delta p \simeq \sigma_c = \sqrt{2D/\lambda}$ in momentum (which is translated by dynamics into a smoothing in position). In our case, $\sigma_c \simeq 0.3$ and $\chi \sim 0.6$, which implies that we are on the border between quantum and classical regimes (defined, respectively, by whether $\sigma_c \chi$ is large or small compared to $\hbar$ [6]). Even though the distributions in Figs. 2(b) and 2(c) are very close, the Wigner function still contains traces of local quantum interference. However, this makes only a minor difference to expectation values, and tends to vanish as the evolution proceeds. We note that, with our choice of parameters, the diffusion term affects the evolution of classical and quantum expectation values roughly to the same extent (Fig. 1). Decoherence destroys the interference pattern in the Wigner function, while, at the same time, noise smooths out the fine structure of the classical distribution

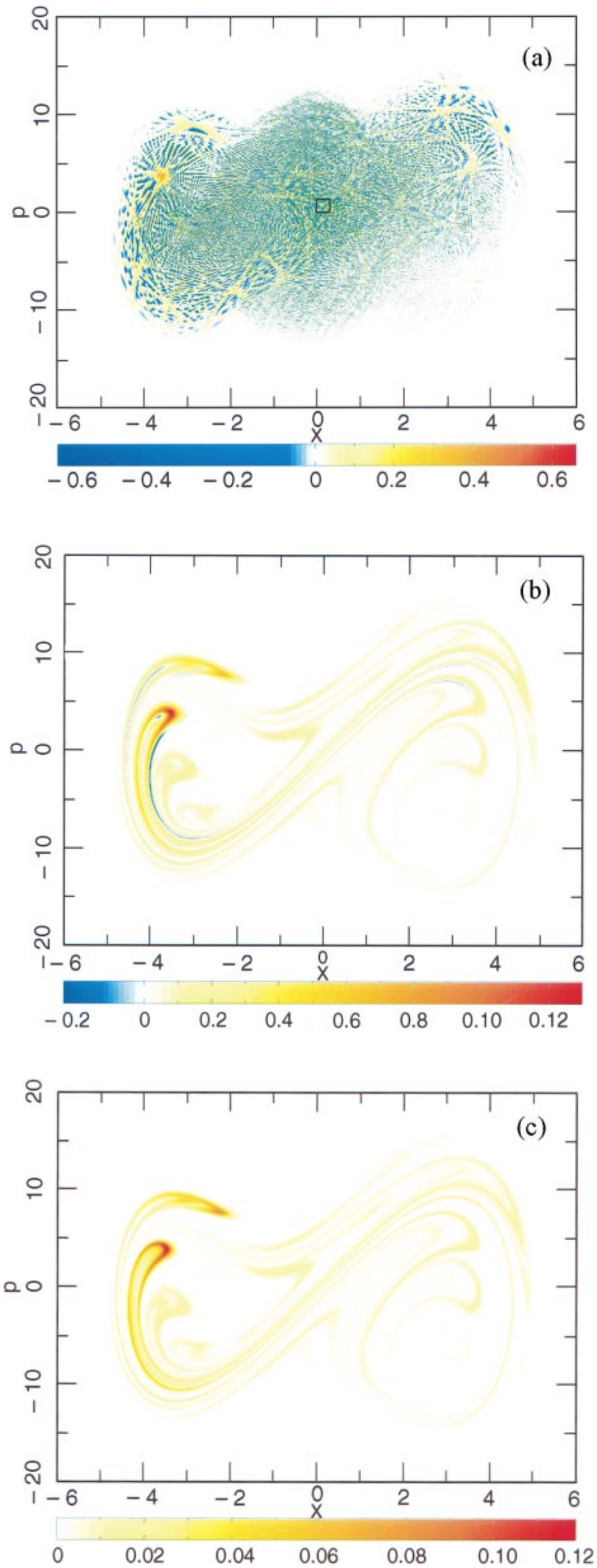

FIG. 2(color). (a) Wigner distribution function from a solution of Eq. (2) at time $t = 8T$, where T is the period of the driving force. The diffusion constant $D = 0$. The box represents a phase space area of $4\hbar$. (b) Wigner distribution function at time $t = 8T$, with diffusion constant $D = 0.025$, illustrating the destruction of large scale quantum coherence. (c) Classical distribution function from a solution of Eq. (3) at time $t = 8T$, with diffusion constant $D = 0.025$.

in such a way that quantum and classical distributions and expectation values both converge to each other. Thus, one concludes that the decohered quantum evolution does go over to the classical Fokker-Planck limit.

In summary, we have provided evidence that in a quantized classically chaotic system, for fixed $\hbar$, classical and quantum expectation values diverge from each other after a time $\sim t_\hbar$. In the case studied here, the discrepancy is $\leq 10\%$ of the typical magnitude of the expectation values. Decoherence was shown to substantially reduce this discrepancy as well as to bring the Wigner and classical distributions very close to each other. In this combined sense, decoherence restores the quantum-classical correspondence. Our results complement previous studies which have focused more on the phase space aspects of the correspondence and the destruction of dynamical localization by noise and dissipation [17].

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