Transition to Normal Fluid Turbulence in Helium II

David J. Melotte and Carlo F. Barenghi
Mathematics Department, University of Newcastle upon Tyne, Newcastle upon Tyne NE1 7RU, England
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Turbulence in the heat flow of helium II manifests itself as a tangle of quantized vortex lines. It has been known for some time that there are two different forms of turbulence, called the T-1 and the T-2 turbulent states. Experiments show that when a critical velocity is exceeded there is a large increase of the superfluid vortex line density and a transition occurs from T-1 to T-2. Until now, the nature of the two turbulent states and of the critical velocity has been a mystery. We present a model which solves this problem by addressing for the first time the issue of the stability of the normal fluid. The computed transition is found to be in good agreement with the observations. [S0031-9007(98)06114-6]

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Some of the most interesting problems in the hydrodynamics of helium II involve quantized vortex lines [1]. Vortex lines appear, for example, when a container filled with helium is set into rotation. In this case, the vortex configuration is ordered: The vortices align along the rotation axis and form a uniform array. There are also situations in which the vortex system is disordered, for example, when the flow of helium is turbulent. Such turbulence, which manifests itself as a *tangle* of quantized vortex lines, is receiving renewed experimental attention: At the University of Lancaster, McClintock and co-workers [2] use intense vortex tangles as models of the early universe, and a superfluid wind tunnel is being built by Donnelly at the University of Oregon [3].

Motivated by this experimental interest in hard turbulence, we address the more basic but still open question of what happens to turbulence at relatively low speed in the much-studied configuration called *counterflow*. Measurements in different apparati and using different techniques clearly show that there exist *two* separate states of turbulence in a circular pipe, called T-1 and T-2 in the literature [4]. T-1 and T-2 are superfluid vortex tangles characterized by very different vortex line densities, T-2's being much larger. If the flow speed exceeds a critical value, there is a transition from T-1 to T-2. The aim of this Letter is to explain for the first time the nature of the two turbulent states and of the critical velocity, until now an outstanding puzzle in the study of superfluid turbulence.

The simplest experimental setup which has been widely used to study the superfluid vortex tangle is a counterflow pipe. One end of the pipe is closed and is provided with a resistor which dissipates a known heat flux W; the other end of the pipe is open to the helium bath. What happens at small W can be easily understood using Landau's two-fluid model. The model describes helium as the intimate mixture of a superfluid component (which flows without any friction) and a normal fluid component (which carries the entropy and viscosity of the liquid). Using subscripts s and n to indicate the super and normal components, respectively, we call ρ_s and ρ_n the densities, $\mathbf{v}_s$ and $\mathbf{v}_n$ the velocities, $\mathbf{j} = \rho_s \mathbf{v}_s + \rho_n \mathbf{v}_n$ the mass flux, $\rho = \rho_s + \rho_n$ helium's total density, T the temperature, and S the entropy. According to Landau's model, the heat flux $W = \rho T S v_n$ is carried by the normal fluid away from the resistor, towards which some superfluid must flow in order to conserve mass. In this way, a relative velocity (thermal counterflow) $v_{ns} = |\mathbf{v}_n - \mathbf{v}_s|$ is set up between the two fluids, which is proportional to the driving heat flux, $v_{ns} = W/\rho_s S T$.

We know from the pioneering experiments of Vinen [5] that if W exceeds a critical value W_{c1} (corresponding to a critical velocity $v_{ns} = v_{c1}$), the frictionless, perfectly heat conducting motion of the superfluid breaks down and a superfluid vortex tangle is created. The vortex line density of the tangle can be easily determined by measuring the attenuation of second sound or from temperature differences. These measurements show that for $v_{ns} > v_{c1}$ the vortex line density L_0 in the tangle is approximately proportional to the square of the driving counterflow velocity,

$$L_0 \approx \gamma^2 v_{ns}^2 , \tag{1}$$

where $\gamma = \gamma(T)$ is some temperature dependent parameter. Geometrically, $L_0^{-1/2}$ represents the average intervortex spacing in the tangle. A major difficulty of interpretation arises, however, because γ varies greatly from experiment to experiment. A very detailed analysis of all experimental data available was carried out by Tough [4], who discovered the existence of various turbulent states characterized by different values of γ. Tough found that, in circular or almost square counterflow channels at increasing values of v_{ns} above the threshold value v_{c1}, there is first a regime of moderate vortex line density, called the T-1 turbulent state. If $W > W_{c2}$ (corresponding to $v_{ns} > v_{c2}$), another state is observed with much larger vortex line density, called T-2. The lower density T-1 state is absent in high aspect ratio (rectangular) channels, in which the line density has essentially the same value as in T-2.

On the theoretical side, the phenomenological theory of Vinen [5] and scaling arguments justified Eq. (1) without determining γ. Later the numerical simulations of Schwarz [6] confirmed (1) and shed more light onto the problem. Schwarz demonstrated that, if $v_{ns} > v_{c1}$, a self-sustaining homogeneous vortex tangle driven by the counterflow is generated by the simple rules of vortex dynamics. Schwarz's theory, unlike Vinen's, has no adjustable parameters and makes quantitative predictions about γ. In applying the theory of Schwarz to the experiments, there are, however, difficulties in understanding what is a homogeneous state of turbulence and what is not. The original calculation of Schwarz assumes that the driving applied normal fluid and superfluid velocity profiles $\mathbf{v}_n$ and $\mathbf{v}_s$ are constant, and therefore so is v_{ns}. Tough [4] found that Schwarz's γ agrees fairly well to the measured values of γ in the T-2 state. Ion trapping experiments [7] probed L_0 across the channel and determined that the superfluid vortex tangle in the T-2 state is spatially homogeneous to within $\frac{1}{10}$ of the channel's walls. This confirms that Schwarz's theory applies to the T-2 state well. Later, an even better agreement was found between the homogeneous theory of Schwarz and measurements in pure superflow [8]. This is expected, because, in pure superflow, $\mathbf{v}_s$ must be constant (the superfluid velocity need not satisfy no-slip boundary conditions), and $\mathbf{v}_n = 0$ by definition. In conclusion, we know little about the T-1 superfluid vortex tangle, because it is not described by Schwarz's theory.

The outstanding questions are the following: Why are there two different kinds of superfluid vortex tangles T-1 and T-2? What is the nature of these tangles? What determines the critical velocity at which there is a transition from T-1 to T-2 with a dramatic increase of the superfluid vortex line density?

The discussion above leads us to conjecture that the existence of T-1 must be linked to some inhomogeneity of $\mathbf{v}_n$ and that the transition at v_{c2} is caused by some instability of the normal fluid. There have been speculations in the literature [4,9] about the nature of $\mathbf{v}_n$, but nobody has ever investigated whether an instability can take place. The existing numerical simulations of Schwarz [6], Samuels and co-workers [10], and deWaele and co-workers [11] have all studied the effects which a prescribed $\mathbf{v}_n$ has on the superfluid tangle. Various forms of $\mathbf{v}_n$ have been considered, ranging from a simple uniform flow to Poiseuille flow, and from a single normal vortex to the vortex tubes of more complex *ABC* flows. The approach which we take here is novel and opposite: We ask what the effect is which the tangle has on the normal fluid itself.

The model which we have developed is the following. According to the observations [12], the channel's length is not critical for the existence of the second superfluid turbulent state T-2, so we consider a cylindrical pipe of infinite length and call R the radius. At velocity $v_{ns} < v_{c1}$, in the regime of thermal counterflow, the normal flow in the pipe is the Poiseuille profile $\mathbf{v}_n = \mathbf{v}_P = V_{ax}(1 - r^2)\hat{\mathbf{z}}$, where the peak velocity V_{ax} is proportional to the prescribed heat flux, $\hat{\mathbf{z}}$ is the unit vector along the pipe, and we have used cylindrical coordinates r, ϕ, z. If $v_{ns} > v_{c1}$, the superfluid field is destabilized and the vortex tangle appears. The tangle affects the normal fluid via the mutual friction force [13]

$$\mathbf{F}_{ns} = \left(\frac{B\rho_n\rho_s}{2\rho}\right) g\,\omega_s(\mathbf{v}_n - \mathbf{v}_s)\,, \tag{2}$$

where B is the mutual friction coefficient [14], $\omega_s = \Gamma L_0$ is the superfluid vorticity, $\Gamma = 9.97 \times 10^{-4}$ cm^2/sec is the quantum of circulation, and g is a constant of order unity which depends on the details for the flow and the tangle. We know little of this tangle, only that its density L_0 corresponds to the less intense T-1 turbulent state. For the sake of simplicity, we assume that $g = \frac{2}{3}$, that is to say the tangle is isotropic. In writing (2), physically we assume that, although the superfluid is turbulent, the mutual friction acts on the vortex lines of the tangle in the same way as it does on vortex lines in the uniform array of lines in rotation. We also neglect the smaller, nondissipative part of the mutual friction force. Equation (2) is a key approximation about the superfluid turbulent state present in a nonuniform flow, and has been recently discussed and tested by Tough [9].

The equations which govern the motion of the normal fluid in the presence of the tangle are

$$\rho_n \frac{\partial \mathbf{v}_n}{\partial t} + \rho_n \mathbf{v}_n \cdot \nabla \mathbf{v}_n = -\frac{\rho_n}{\rho}\nabla p - \rho_s S \nabla T + \mu \nabla^2 \mathbf{v}_n - \mathbf{F}_{ns}\,, \tag{3}$$

and the incompressibility condition $\nabla \cdot \mathbf{v}_n = 0$, where μ is helium's viscosity. The experiments indicate that the T-1 state is macroscopically [15] steady and that it exists at Reynolds numbers of the normal fluid in the range from approximately 20 to 200, which is at least 1 order of magnitude below the transition to turbulence in classical pipe flow [16]. It is therefore fair to assume that the solution of the normal fluid [Eq. (3)] in the T-1 state is some time-independent laminar function $\mathbf{v}_n^0$; correspondingly, the superfluid profile is some other function $\mathbf{v}_s^0$. Our aim is to determine the stability of the normal fluid $\mathbf{v}_n^0$ under the increasing forcing due to the tangle at higher and higher values of L_0. To achieve the aim, we use standard linear stability theory.

We perturb the basic state $\mathbf{v}_n^0$ by introducing small disturbances which are proportional to a small parameter ϵ. Then we enforce the condition that the total velocity $\mathbf{v}_n^0 + \epsilon\mathbf{v}_n'$ is a solution to Eq. (3), neglect terms which are proportional to ϵ^2, and subtract the equation satisfied by the basic state $\mathbf{v}_n^0$. In order to compare results from different experiments, it is convenient to write the resulting linearized equation for the perturbations $\mathbf{v}_n'$ in dimensionless form. To do so, we scale distances with the radius R of the pipe and time with the viscous diffusion time scale R^2/ν_n, where $\nu_n = \mu/\rho_n$ is the normal fluid's kinematic viscosity. We have then

$$\frac{\partial \mathbf{v}_n'}{\partial t} + \mathbf{v}_n' \cdot \nabla \mathbf{v}_n^0 + \mathbf{v}_n^0 \cdot \nabla \mathbf{v}_n' = -\nabla p' + \nabla^2 \mathbf{v}_n' - \beta \mathbf{v}_n'\,, \tag{4}$$

where

$$\beta = \frac{B\rho_s\Gamma}{3\rho\nu_n} L_0 R^2, \tag{5}$$

and p' is the rescaled efficient pressure. Equation (4) must be solved together with $\nabla \cdot \mathbf{v}_n' = 0$ and the no-slip boundary conditions $\mathbf{v}_n' = 0$ at $r = 1$. We have no direct information about $\mathbf{v}_n^0$, but we argue that in the first approximation $\mathbf{v}_n^0$ will be rather similar to the Poiseuille profile $\mathbf{v}_P$. The argument is the following. Although we do not know the precise shape of the superfluid velocity $\mathbf{v}_s^0$, we can use the counterflow condition $\mathbf{j} = 0$ to find an approximate average value of $\mathbf{v}_s^0$ in terms of $\mathbf{v}_n^0$ and then solve (3). In the parameter regime of interest here, we find that the solution $\mathbf{v}_n^0$ differs only slightly from the parabolic profile $\mathbf{v}_P$, the only difference being a mild flattening of the shape [17]. It is therefore justified to use $\mathbf{v}_P$ as the approximate basic state which we perturb in the linear stability calculation. A similar approximation to the basic state is used, for example, in the study of the stability of the flow between two rotating concentric cylinders, when one determines the appearance of Taylor vortex flow by perturbing the azimuthal Couette velocity profile while neglecting the small axial and radial flow induced by the ends.

To solve Eq. (4) we make the standard assumption that the perturbations have the dependence $\exp(\sigma t + im\phi + ikz)$. In this way, we obtain an eigenvalue problem for the growth rate σ as a function of the dimensionless axial flow V_{ax}, the forcing parameter β, and the axial and azimuthal wave numbers k and m. If $\mathrm{Re}(\sigma) > 0$, the basic flow $\mathbf{v}_n^0$ is unstable. Note that, if $\beta = 0$, the problem which we consider reduces to an Osborne Reynolds' classical problem of the stability flow in a cylindrical pipe [15]; in this case, we know that the flow is linearly stable, and the onset of turbulence which is observed in the experiments is determined by finite amplitude disturbances which can vary from one experiment to the other. The eigenvalue problem is solved by using a Chebyshev collocation method [18]. We expand the perturbations over spectral functions which satisfy both the boundary conditions and the regularity conditions on the axis (the latter depending both on the particular variable in consideration and on the azimuthal wave number). The spectral functions are built using suitable combinations of Chebyshev polynomials $T_j(r)$ of degree j. The spectral expansions are truncated after a sufficiently high number N of terms and tested for spectral accuracy. Typically N ranges from 20 to 30. The resulting equations for the unknown coefficients in the spectral expansions form a matrix eigenvalue problem which is solved using a NAG (Numerical Algorithms Group, Inc.) routine.

We explore the parameter space and look for the marginal states, which are defined by the condition that $\sigma = 0$. We find that axisymmetric perturbations ($m = 0$) are always stable, but nonaxisymmetric modes ($m \neq 0$) can become unstable if β is large enough [19]. At any value of V_{ax}, if we increase β, the first mode to become unstable is $m = 1$, followed by $m = 2$, $m = 3$, etc. We consider now the mode $m = 1$. For a given value of k, we find that there is a threshold value $\beta = \beta_{\min}$ below which the flow is always stable no matter what the value of V_{ax} is; at $\beta = \beta_{\min}$ a particular axial flow V_{ax} can destabilize the perturbation of that particular wave number k. We find that the smaller k is, the larger the required V_{ax} becomes (see Fig. 1). It is apparent from Fig. 1 that below a critical value $\beta \approx 13$ no perturbation will grow. The critical value is slightly larger, 14 rather than 13, if $V_{ax} < 10$, but in the first approximation we take $\beta_{c2} = 13$ as the stability boundary.

From β_{c2} and Eq. (5), we compute the quantity $\Delta = L_0^{-1/2} R^{-1}$ that is the critical dimensionless tangle intensity which destabilizes the normal fluid. Geometrically, Δ is the average vortex line spacing expressed in units of the pipe's radius. Note that, despite its simple geometrical meaning, Δ does not depend on T because of the complicated temperature dependence of the terms which enter the definition of β.

Figure 2 compares our Δ to experimental measurements [20] of the transition from the state T-1 to the state T-2 in cylindrical pipes. The quantitative agreement between theory and experiments is very good, more so if one considers the relative simplicity of our model. It is not clear how much significance should be attached to the slight difference between the temperature dependence of Δ and the observations. This dependence arises from the various terms on the definition of β: Some of these terms, individually, change rapidly with temperature, but, taken together, almost compensate each other, so β is sensitive on the individual values. The large scatter of the data suggests that more precise experiments are needed. For each experimental data set, the pipe's diameter $D = 2R$ and the critical Reynolds numbers Re_{c2} of the normal fluid at the transition are indicated in the figure caption, where $\mathrm{Re}_{c2} = v_n R/\nu_n$, $v_n = W_{c2}/\rho ST$, and W_{c2} is the observed critical heat flux. It is apparent that the agreement between theory and observations covers a wide range of parameters.

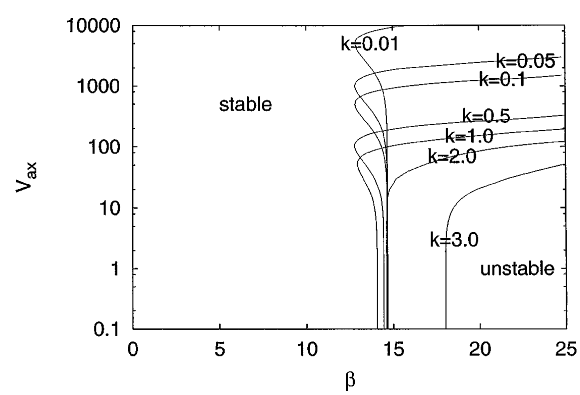

FIG. 1. Stability boundary of $m = 1$ for different values of axial wave number k. The basic state is stable on the left of each curve and unstable to the right.

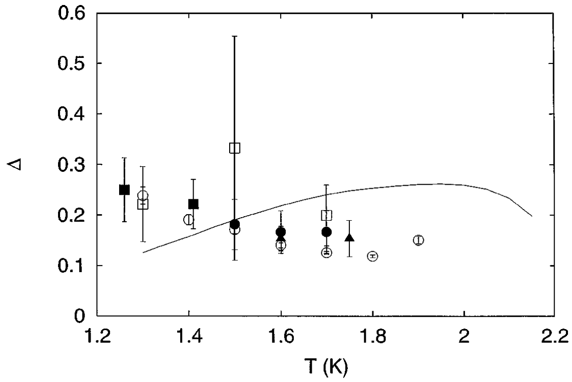

FIG. 2. Transition between T-1 and T-2 states, comparison between theory and experiments [20]. Solid line: theoretical prediction; white circles: Ladner *et al.*, $D = 0.0129$ cm, Re_{c2} ranging from 35 to 131; black circles: Martin and Tough, $D = 0.1$ cm, Re_{c2} from 122 to 186; white squares: de Goeje and van Beelen, $D = 0.139$ cm, Re_{c2} from 26 to 272; black squares: Marees and van Beelen, $D = 0.0216$ cm, Re_{c2} from 33 to 61; white triangle: Marees and van Beelen, $D = 0.0133$ cm, $\mathrm{Re}_{c2} = 45$; black triangles: Griswold *et al.*, $D = 0.0132$ cm, Re_{c2} from 111 to 128.

The good agreement between theory and experiments confirms that the basic physical mechanism of the T-1 to T-2 transition is an instability of the normal fluid: As the applied heat current increases, the increasingly dense vortex tangle leads to an instability of the laminar flow of the normal fluid via bulk friction proportional to the vortex line density L_0. A natural future development of our theory would be to include the back reaction of the vortex tangle on the instability itself, taking into account more details of the structure and the dynamics of the tangle beyond its density L_0. This development may clarify the temperature dependence. The difficulty to overcome is the lack of a governing equation for the turbulent superfluid: Further modeling and the introduction of unknown parameters may be required. One possibility may be to use Vinen's dynamical equation [4] for L_0. The alternative is a full scale 3-dimensional numerical simulation of *both* the vortex tangle and the turbulent normal fluid, but it would be an immense computational task.

In conclusion, we have addressed for the first time the key issue of the stability of the normal fluid. The picture which results from our calculation is rather simple. In the state T-1, the superfluid is turbulent, but the density L_0 of the superfluid vortex tangle is not sufficient to alter significantly the laminar profile of the normal fluid. If we increase v_{ns}, eventually the line density L_0 becomes large enough that at $v_{ns} = v_{c2}$ a transition takes place and the normal fluid is destabilized. The linear stability approach cannot predict what happens to the normal fluid past the instability. But our model is closely related to the classical pipe flow problem, so we should expect that the normal fluid becomes turbulent. This is supported by the experimental observation that there is a large increase of the dissipation for $v_{ns} > v_{c2}$. The T-2 state therefore corresponds to turbulence of *both* the superfluid and the normal fluid components. This is consistent with the good agreement between Schwarz's calculations and the measurements in the T-2 state: The uniform profile $\mathbf{v}_n$ used by Schwarz corresponds to the uniform turbulent profile in the pipe.

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