Is There a Landau Pole Problem in QED?

M. Göckeler,[1] R. Horsley,[2] V. Linke,[3] P. Rakow,[4] G. Schierholz,[4,5] and H. Stüben[6]
[1]Institut für Theoretische Physik, Universität Regensburg, D-93040 Regensburg, Germany
[2]Institut für Physik, Humboldt-Universität zu Berlin, D-10115 Berlin, Germany
[3]Institut für Theoretische Physik, Freie Universität Berlin, D-14195 Berlin, Germany
[4]Deutsches Elektronen-Synchrotron DESY, HLRZ and Institut für Hochenergiephysik, D-15735 Zeuthen, Germany
[5]Deutsches Elektronen-Synchrotron DESY, D-22603 Hamburg, Germany
[6]Konrad-Zuse-Zentrum für Informationstechnik Berlin, D-14195 Berlin, Germany
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We investigate a lattice version of QED with four flavors by numerical simulations. For the renormalized charge (e_R) and mass we find results which are consistent with e_R vanishing in the continuum limit. A detailed study of the relation between bare and renormalized quantities reveals that the Landau pole lies in a region of parameter space which is made inaccessible by spontaneous chiral symmetry breaking. [S0031-9007(98)06002-5]

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QED is the best tested of all quantum field theories. But all its success is in the context of perturbation theory. It has long been known that there are potential problems in the foundations of the theory due to the existence of the so-called Landau pole [1]. In the leading logarithmic calculation one finds

$$\frac{1}{e_R^2} - \frac{1}{e^2} = \beta_1 \ln \frac{\Lambda}{m_R}, \qquad \beta_1 = \frac{N_f}{6\pi^2}, \tag{1}$$

where e (e_R) is the bare (renormalized) charge, m_R is the renormalized fermion mass, N_f is the number of flavors, and Λ is the ultraviolet cutoff. When one attempts to send the cutoff to infinity while keeping e_R fixed, one finds that e diverges at

$$\Lambda = \Lambda_L \equiv m_R e^{1/\beta_1 e_R^2}, \tag{2}$$

the location of the Landau pole. The problem can also be seen by looking at the gauge invariant part of the photon propagator

$$\frac{D(k)}{k^2} = \frac{1}{k^2[1 - (\beta_1/2)\ln(k^2/m_R^2)]}, \tag{3}$$

which has a ghost pole at $k^2 = \Lambda_L^2$. This would mean that the entire theory is applicable only for momenta smaller than Λ_L. On the other hand, when one keeps e fixed and sends the cutoff to infinity, the renormalized charge goes to zero, meaning that the theory is trivial. The situation in two-loop perturbation theory is much the same. The (renormalized) β function

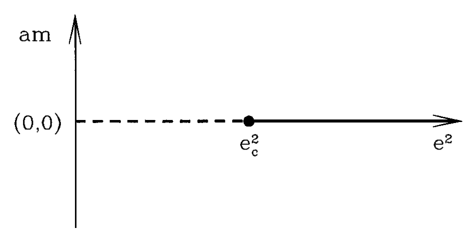
$$\beta_R \equiv m_R \frac{\partial e_R^2}{\partial m_R}\bigg|_{e^2} = \beta_1 e_R^4 + \beta_2 e_R^6 + \ldots \tag{4}$$

remains positive for all e_R^2, and the Landau pole is displaced to lower values

$$\Lambda_L = m_R e^{1/\beta_1 e_R^2}\left(\frac{\beta_2 e_R^2}{\beta_1 + \beta_2 e_R^2}\right)^{\beta_2/\beta_1^2}. \tag{5}$$

QED is not the only theory with a Landau pole problem. Every theory which is not asymptotically free suffers from this problem. While $\Lambda_L \simeq 10^{227}$ GeV if only the electron is considered, $\Lambda_L \simeq 10^{34}$ GeV in the standard model. In the minimal supersymmetric standard model (MSSM) $\Lambda_L \simeq 10^{20}$ GeV, and in the MSSM with four Higgses, which offers a solution to the strong CP problem, the Landau pole moves down to $\Lambda_L \simeq 10^{17}$ GeV [2]. Thus the Landau pole is by no means academic.

To find a solution to this problem, one must consider a nonperturbative formulation of QED. Thus it is natural to investigate the problem on the lattice. On the lattice the inverse lattice spacing takes over the role of the ultraviolet cutoff, $a^{-1} \sim \Lambda$. Early calculations have shown that the noncompact formulation of the theory using staggered fermions undergoes a second order chiral phase transition at strong coupling [3,4]:



The solid line $am = 0$, m being the bare mass, $e^2 > e_c^2$ is a line of first order chiral phase transitions, where am_R, $\sigma \equiv a^3\langle\bar{\psi}\psi\rangle \neq 0$, even though the bare mass is zero. The dashed line $am = 0$, $e_c^2 > e^2$ is a line of second order phase transitions on which am_R, $\sigma = 0$. A meaningful continuum limit can be taken at the tricritical point $am = 0$, $e^2 = e_c^2$, because here we can take a to zero while keeping m_R fixed.

To understand the continuum limit of the theory, we need to know the renormalized charge as a function of the cutoff in the critical region. We have recently computed the chiral condensate on large lattices [5]. In this Letter we compute am_R and e_R with the aim to understand the fate of the Landau pole in QED.

Fermion mass.—Let us begin with the renormalized mass. We obtain am_R from the fermion propagator as outlined in [6]. We are using staggered fermions which in the continuum limit correspond to $N_f = 4$ flavors of dynamical Dirac fermions. The results are shown in Fig. 1.

We have made the phenomenological observation in [6] that the chiral condensate σ is a function of am_R alone in the critical region, and in [5] we have found an equation of state (EOS) which describes σ very well. Therefore we choose to express am_R indirectly in terms of $\sigma(e^2, am)$ rather than directly as a function of e^2, am. In Fig. 2 we plot σ against am_R. We see that the data from all e lie on the same curve. We find that σ is well described by a polynomial. Combining the EOS with the polynomial gives the curves shown in Fig. 1 and the extrapolation to $am = 0$. (Here we have used fit 1 of [5]. Our results do not change qualitatively if we use any of the other fits described there.) For $1/e^2 < 1/e_c^2$ chiral symmetry is broken, and even at $am = 0$ the renormalized mass is nonzero. This means there is an excluded region shown in white in Fig. 1 (the accessible region being shown in gray).

Renormalized charge.—The renormalized charge is obtained from the residue of the photon propagator, $e_R^2 = Z_3 e^2$ and $Z_3 = \lim_{k\to 0}\lim_{V\to\infty} D(k)$. We can compute $D(k)$ on the lattice, but not at $k = 0$. The smallest momentum that we can reach is $2\pi/aL$, where L is the lattice size ($L = 16$ and 12 in our case). To extrapolate to $k = 0$ we need to make a fit to the photon propagator. The k dependence of the photon propagator is given by

$$\frac{1}{e^2 D(k)} - \frac{1}{e^2} = -\Pi(k, m_R, L)\,, \qquad (6)$$

where Π is the polarization function. In the infinite volume limit we then have

$$\frac{1}{e_R^2} - \frac{1}{e^2} = -\Pi(0, m_R, \infty)\,. \qquad (7)$$

We have already seen that the nonperturbative Π is actually very close to the result of one-loop renormalized perturbation theory [6]. So it is reasonable to make an ansatz which is inspired by renormalization group improved two-loop perturbation theory. In [7] it is shown that to next-to-leading logarithmic order the polarization function can be written

$$\Pi = U - \frac{V}{U}\ln(1 - e^2 U)\,, \qquad (8)$$

where U is the one-loop perturbative result, and V the two-loop one. The lattice result for U is known [6]. For V we make the ansatz

$$V = v_0 + v_1 U\,. \qquad (9)$$

This is motivated by the small k^2 and m_R^2 limits. For $a^2 m_R^2 \ll a^2 k^2 \ll 1$ we should have $V \simeq (\beta_2/2)\ln a^2 k^2$, and for $a^2 k^2 \ll a^2 m_R^2 \ll 1$ we should have $V \simeq (\beta_2/2)\ln a^2 m_R^2$. The one-loop result U has these properties. We fit this ansatz to a total of 52 photon propagators on 16^4 and 12^4 lattices for various values of am, e^2 in the range $0.005 \le am \le 0.16$, $0.17 \le 1/e^2 \le 0.22$ close to the critical point at $1/e_c^2 = 0.190\,40(9)$ [5]. A plot for one particular parameter set is shown in Fig. 3. For the fit parameters we obtain $v_0 = -0.002\,07(2)$ and $v_1 = -0.032\,8(7)$, giving

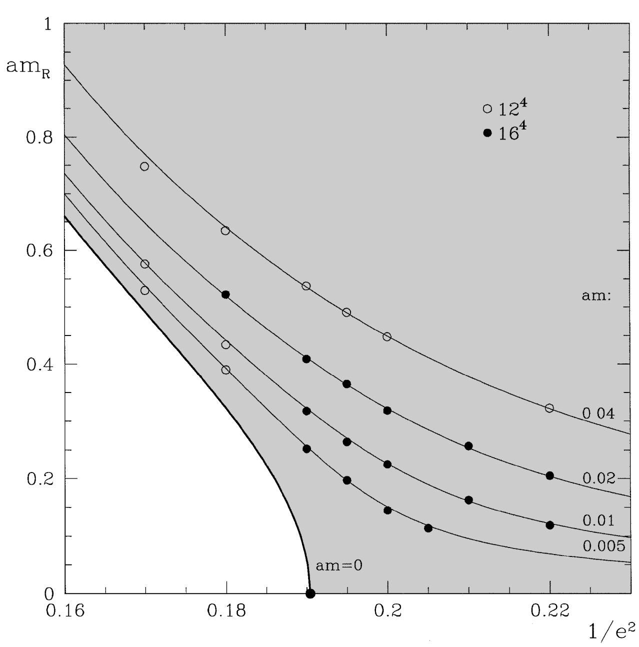


FIG. 1. The renormalized mass against the bare coupling on 12^4 and 16^4 lattices in the critical region.

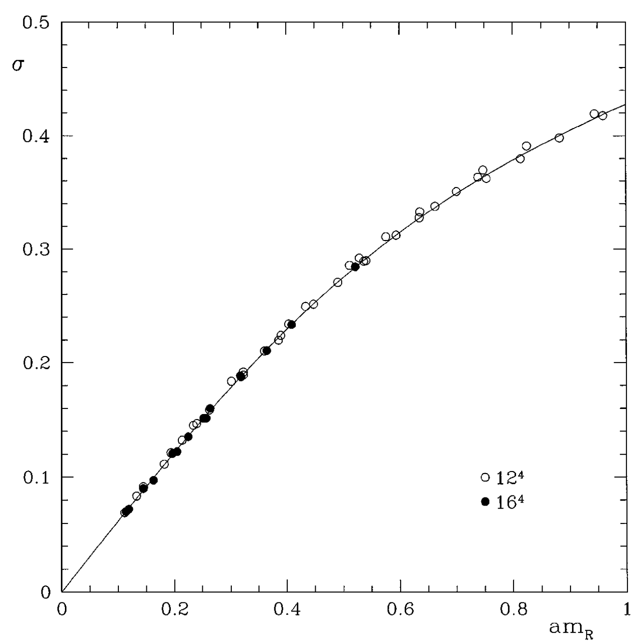


FIG. 2. The chiral condensate against the renormalized mass on 12^4 and 16^4 lattices for $0.17 \le 1/e^2 \le 0.22$ and $0.005 \le am \le 0.16$. The curve is a polynomial fit.

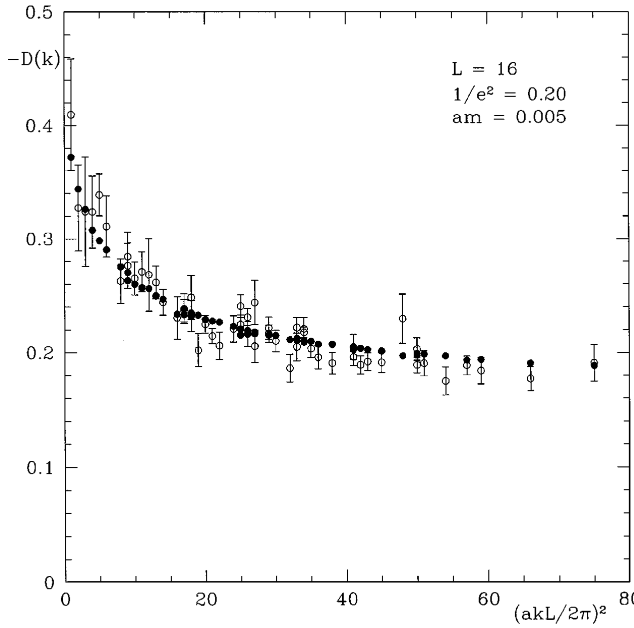


FIG. 3. The residue of the photon propagator against the momentum for $1/e^2 = 0.20, am = 0.005$ on the 16^4 lattice. The open symbols are the data, the solid symbols are the fit.

χ^2/d.o.f. = 1.7. Two-loop continuum perturbation theory would give $v_1 \equiv \beta_2/\beta_1 = 3/16\pi^2 = 0.0190$. In Fig. 4 we show the resulting β function for $e^2 = e_c^2$. We compare this with the one-loop result. We see that the β function is a little smaller than the one-loop value and is positive. In particular this means that there is no ultraviolet stable zero in the β function out to $e_R^2 = e_c^2$, the maximal value e_R^2 can take because $Z_3 \leq 1$ [8,9]. As $am_R \rightarrow \infty$ fermion loops are suppressed and $e_R^2 \rightarrow e^2$, so that the β function vanishes. But this is of course not an interesting zero of the β function.

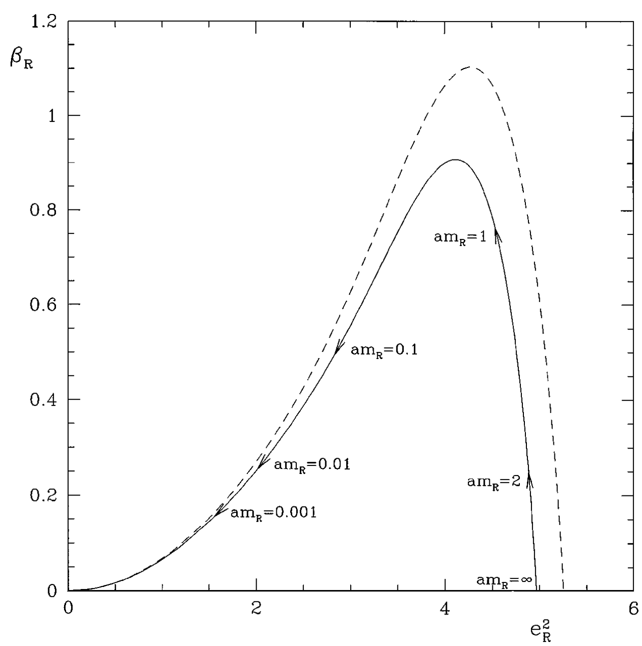


FIG. 4. The β function against the renormalized charge. The solid curve is our result; the dashed curve is the lattice one-loop result.

The Landau pole.—Having calculated the renormalized mass and charge, we are now able to discuss the mapping from the bare parameters am, e to the renormalized parameters am_R, e_R. Qualitatively this is displayed in Fig. 5. One can choose any $e^2 \geq 0$, am shown in the top part of the figure by the gray region. This is then mapped onto the corresponding gray region in the bottom part of the figure. The line $am = 0$, $0 \leq e^2 \leq e_c^2$ is mapped onto the point $am_R = e_R^2 = 0$. For $e^2 > e_c^2$ we have already seen in Fig. 1 that $am_R > 0$, even when $am = 0$, because of chiral symmetry breaking. Thus the line $am = 0$, $e_c^2 \leq e^2 \leq \infty$ is mapped onto the border line of the gray regions, and the white area is inaccessible for any combination of bare parameters. From this figure we

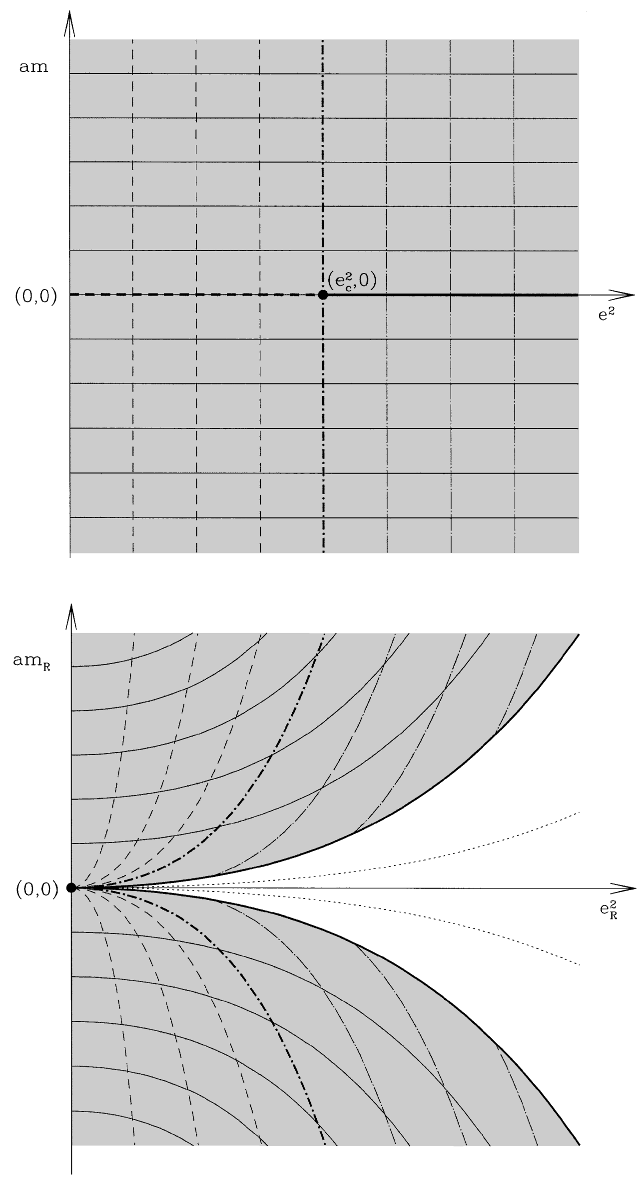


FIG. 5. A sketch of the mapping from the bare parameter plane (top) to the renormalized parameter plane (bottom).

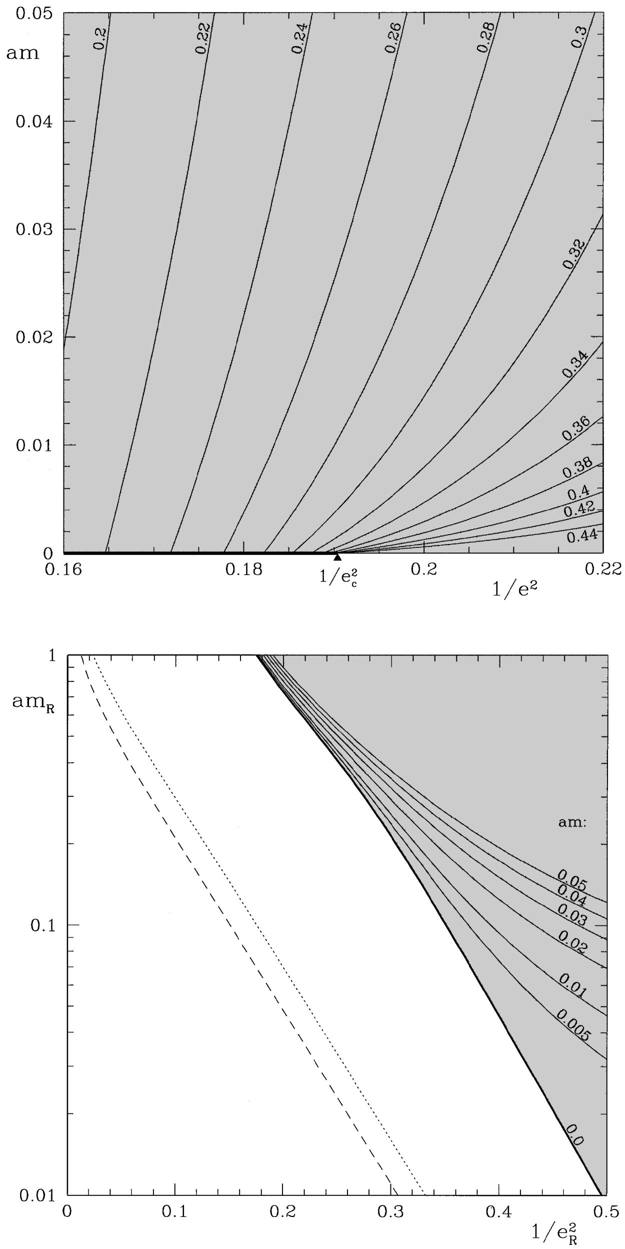


FIG. 6. The (quantitative) mapping of the bare parameter plane (top) to the renormalized parameter plane (bottom).

evidently have triviality. Removing the cutoff ($a \longrightarrow 0$) is only possible at $e_R^2 = 0$. At any finite value of e_R^2 there is a minimal possible value for $|am_R|$, namely the boundary of the gray region. The position of the Landau pole is sketched by the dotted line.

We now turn to the quantitative analysis of the problem. In Fig. 6 we plot again the bare and renormalized planes, this time using $1/e^2$ and $1/e_R^2$, respectively, as the horizontal variables because this displays the asymptotic behavior best. The curves are lines of constant $1/e_R^2$ (top part) and lines of constant am (bottom part), respectively. All lines of constant e_R^2 end on the first order phase transition line, and only the line $e_R^2 = 0$ goes into the critical point. This is another expression of triviality of the theory. In the bottom part of the figure the gray region is again the allowed region, and the white region is inaccessible. The border line is the line $am = 0$. From Eq. (7) we find the Landau pole by setting the bare charge to infinity. This gives the dotted line. We see that it completely lies in the inaccessible region. It runs roughly parallel to the border line $am = 0$.

We also want to be sure that the photon propagator has no ghost pole for any k^2. Beyond leading logarithmic order the extra pole in the photon propagator and the divergence in e^2 need not appear at the same place. The ghost pole position (in the infinite volume) is given by $1/k^2D(k) = 0$. It has a solution if

$$\frac{1}{e_R^2} < \max_k \Pi(k, m_R, \infty) - \Pi(0, m_R, \infty). \qquad (10)$$

The solution is given by the dashed line in the bottom part of Fig. 6. This lies close to the Landau pole, even deeper in the inaccessible region.

In conclusion, from Figs. 5 and 6 we see that the triviality of QED is intimately connected with chiral symmetry breaking. Any attempt to remove the cutoff is always thwarted by the dynamically generated fermion mass. In particular this means that spinor QED (with four flavors) does not exist as an interacting theory, similar to what Coleman and Weinberg [10] found for scalar QED.

We have restricted our analysis to pure QED. If there were a non-Gaussian fixed point there might be other relevant or marginal operators which could be added to the action. This is a subject for further study.

We have also seen that chiral symmetry breaking allows QED to escape the Landau pole problem. While the bare parameters of the theory can take any values, the renormalized parameters are restricted. The Landau pole and ghost problem only occur deep in the inaccessible e_R^2, am_R region. Chiral symmetry breaking is always strong enough to push the Landau pole above the cutoff.

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