Analog Quantum Error Correction

Seth Lloyd[1] and Jean-Jacques E. Slotine[1,2]
[1]*d'Arbeloff Laboratory for Information Systems and Technology, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139*
[2]*Nonlinear Systems Laboratory, Department of Mechanical Engineering, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139*
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Quantum error-correction routines are developed for continuous quantum variables such as position and momentum. The result of such analog quantum error correction is the construction of composite continuous quantum variables that are largely immune to the effects of noise and decoherence. [S0031-9007(98)05861-X]

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The quantum systems used for quantum computation and quantum communications are small, sensitive, and easily perturbed [1–8]. The theory of quantum error-correcting codes provides a new set of techniques for protecting quantum systems against the effects of noise and decoherence [9–29]. Conventional quantum error-correcting codes are effective only for discrete variables, however. This Letter presents a set of *analog* quantum error-correcting routines that protect continuous variables such as position and momentum against noise and decoherence. These error-correcting routines can, in principle, be enacted using simple Hamiltonian operations to stabilize the states of arbitrary continuous quantum variables. Particular applications include error correction for quantum communications using continuous variables such as photon momentum, and for analog quantum computers used for simulating continuous quantum systems [30–31].

The simplest classical discrete error-correcting routine is triple modular redundancy, in which three bits are initially set to the same value and checked at regular intervals to see if they still have the same value: If one of them differs, it is reset to the value of the two others. If the error rate per bit per unit time is λ, then performing this "voting" routine at intervals of time $\delta t \ll 1/\lambda$ results in a new error rate of $3\lambda^2\,\delta t \ll \lambda$.

The discrete error-correcting technique of triple modular redundancy can be adapted to continuous quantum variables. Consider three continuous "position" quantum variables with states $|x_1x_2x_3\rangle_{123}$, and errors corresponding to unitary operators $e^{-iQ(P_j)}$, where $P_j = -i\partial/\partial x_j$ is the "momentum" operator on the jth variable and Q is a polynomial function of P_j (we call these variables position and momentum for convenience only: the method works for any continuous variable and its conjugate). Such an error takes

$$|x\rangle_i \rightarrow e^{-iQ(P_j)}|x\rangle_j = (1/\sqrt{2\pi})\int_{-\infty}^{\infty} e^{-ipx-iQ(p)}|p\rangle_j\,dp\,, \tag{1}$$

where $|p\rangle_j = (1/\sqrt{2\pi})\int_{-\infty}^{\infty} e^{ipx}|x\rangle_j\,dx$. The error acts on only one variable: $|x\rangle_k \rightarrow |x\rangle_k$ for $k \neq j$. For example, $Q(P_j) = \delta x\, P_j$ takes

$$|x\rangle_j \rightarrow (1/\sqrt{2\pi})\int_{-\infty}^{\infty} e^{-ipx-ip\delta x}|p\rangle_j\,dp = |x + \delta x\rangle_j\,. \tag{2}$$

To correct for these errors, apply the following quantum "continuous voting" procedure. We assume that a variable can be prepared in the state $|0\rangle_j$ by some dissipative process such as cooling, and that the state $|x\rangle_j$ can also be prepared, e.g., by applying the displacement Hamiltonian $\eta x P_j$ to the state $|0\rangle_j$ for a time $1/\eta$. To "vote," apply the following procedure to three continuous quantum variables (for example, the x, y, and z components of the position of a single particle in three dimensions), initially in the state $|xxx\rangle_{123}$, together with three ancilla variables $|x_1x_2x_3\rangle_{1'2'3'}$, initially in the state $|000\rangle_{1'2'3'}$: (i) Suppose that an error occurs to one of the variables, e.g., the second one:

$$|x\rangle_2 \rightarrow e^{-iQ(P_2)}|x\rangle_2 = (1/\sqrt{2\pi})\int_{-\infty}^{\infty} e^{-ipx-iQ(p)}|p\rangle_2\,dp$$
$$\equiv \int_{-\infty}^{\infty}\alpha(x,x')\,|x'\rangle_2\,dx', \tag{3}$$

where $\alpha(x,x') = (1/\sqrt{2\pi})\int_{-\infty}^{\infty} e^{-ip(x-x')-iQ(p)}\,dp$. Reprepare the ancilla variables in the state $|000\rangle_{1'2'3'}$ (this corrects any error that has occurred to the ancillae). The overall state of the variables and the ancillae is now

$$(|x\rangle_1|0\rangle_{1'})\left(\int_{-\infty}^{\infty}\alpha(x,x')\,|x'\rangle_2\,dx'|0\rangle_{2'}\right)(|x\rangle_3|0\rangle_{3'})\,. \tag{4}$$

(ii) Perform a continuous quantum analog of voting. We will assume that we can perform simple real-number operations such as comparing the values of two variables to see if they are equal, and adding the value of one variable to another. So, for example, we will assume that we can perform operations such as comparing $|x_1\rangle_1$ and $|x_2\rangle_2$ to see if $x_1 = x_2$ and, if they are, performing operations such as $|x_1\rangle_1|x_2\rangle_2|x_3\rangle_3 \rightarrow |x_1\rangle_1|x_2\rangle_2|x_3 + x_1\rangle_3$. Such operations are reversible and correspond to unitary transformations on Hilbert space. They can be accomplished by the application of simple interactions between variables. For example, the conditional addition operation just described can be accomplished by applying the Hamiltonian $\eta\Delta(X_1, X_2)X_1P_3$ for time $1/\eta$, where $\Delta(X_1, X_2) = \int_{-\infty}^{\infty} |x\rangle_1\langle x| \otimes |x\rangle_2\langle x|\,dx$. Such an operation can be thought of as a continuous version of a quantum logic gate. (In real life, all such operations can be performed only to finite precision; we will assume infinite precision for the moment and discuss the effects of finite precision below.) If only one error has occurred, then two of the x's are always equal. So, one by one, compare each of the $|x\rangle_i$ to the other two. Let ijk be some permutation of 123. If $x_i = x_j = x$, then add the value of $x_k - x$ to the ancilla state $|0\rangle'_k$. If $x_i \neq x_j$, then do nothing. In our case, only $|x\rangle_2$ and its ancilla will be affected. The variables and ancillae are now in the state

$$(|x\rangle_1|0\rangle'_1)\left(\int_{-\infty}^{\infty}\alpha(x,x')\,|x'\rangle_2|x'-x\rangle_{2'}\,dx'\right)(|x\rangle_3|0\rangle_{3'})\,. \tag{5}$$

(iii) Now, if $x_i = x_j$, subtract the value of the kth ancilla variable from the original kth variable, leaving the state

$$(|x\rangle_1|0\rangle_{1'})\left(|x\rangle_2\int_{-\infty}^{\infty}\alpha(x,x')\,|x'-x\rangle_{2'}\,dx'\right)(|x\rangle_3|0\rangle_{3'})\,. \tag{6}$$

Substituting in the explicit expression for $\alpha(x,x')$ given above allows this state to be written as

$$\begin{aligned}(|x\rangle_1|0\rangle_{1'})&\left(|x\rangle_2(1/2\pi)\int_{-\infty}^{\infty}e^{-ip(x-x')-iQ(p)}|x'-x\rangle_{2'}\,dx'\,dp\right)(|x\rangle_3|0\rangle_{3'})\\ &= (|x\rangle_1|0\rangle_{1'})\left(|x\rangle_2(1/\sqrt{2\pi})\int_{-\infty}^{\infty}e^{-iQ(p)}|p\rangle_{2'}\,dp\right)(|x\rangle_3|0\rangle_{3'})\\ &= (|x\rangle_1|0\rangle_{1'})\,(|x\rangle_2e^{-iQ(P_{2'})}|0\rangle_{2'})\,(|x\rangle_3|0\rangle_{3'})\,.\end{aligned} \tag{7}$$

The error has now been corrected.

This procedure corrects the error by restoring the three variables to the original continuous "code word" $|xxx\rangle_{123}$ while leaving the ancilla in a state that is independent of the initial value of x. The fact that the ancilla is in a state that depends only on error operator $e^{-iQ(P_i)}$ applied and not on the particular code word to which it is applied means that the procedure restores not only continuous code words but arbitrary superpositions of the code words $\int_{-\infty}^{\infty}\psi(x)\,|xxx\rangle\,dx$.

To continue correcting errors, simply return the ancilla variables to $|000\rangle_{1'2'3'}$ and apply the procedure again a time δt later. Just as in classical triple modular redundancy, performing the error-correcting routine at intervals δt reduces the error rate from λ to $3\lambda^2\,\delta t$, which can be made as small as desired by decreasing δt. As an example, consider the case where $|xxx\rangle_{123}$ corresponds to the position of a free particle in three dimensions, the ancilla corresponds to a second particle initially located at the origin, and the "error" operator is supplied by the particle's natural Hamiltonian. Here, the error-correcting routine suppresses to first order in δt the dispersive spreading of the particle's wave packet away from the line $x = y = z$ while enhancing the dispersion of the ancilla wave packet.

It can be seen easily by interchanging the roles of x and p above that continuous code words of the form $|ppp\rangle_{123}$ can be protected against arbitrary errors of the form $e^{iR(X_j)}$, where X_j is the position operator on the jth variable and R is a polynomial function of X_j. In analogy to the $|xxx\rangle$ error-correcting routine, we assume that variables and ancillae can be prepared in momentum eigenstates $|p=0\rangle_j$ and that states $|p\rangle_j$ can be created by applying the "boost" Hamiltonian ηpX_j to the state $|p=0\rangle_j$ for a time $1/\eta$.

The following algorithm corrects both phase and displacement errors. Define the state

$$|\mathbf{p}\rangle_{123} \equiv (1/\sqrt{2\pi})\int_{-\infty}^{\infty}e^{i\mathbf{p}x}|xxx\rangle_{123}\,dx\,. \tag{8}$$

Such a state can be created from the state

$$|\mathbf{p}\rangle_1|0\rangle_2|0\rangle_3 = (1/\sqrt{2\pi})\int_{-\infty}^{\infty}e^{i\mathbf{p}x}|x\rangle_1|0\rangle_2|0\rangle_3\,dx$$

by applying the Hamiltonian ηX_1P_j for time $1/\eta$ to effect the unitary operation $|x\rangle_1|y\rangle_j \rightarrow |x\rangle_1|x+y\rangle_j$, for $j = 2, 3$.

The error operator $e^{iR(X_j)}$ has the same effect on the triple-variable state $|\mathbf{p}\rangle_{123}$ that it has on the single-variable state $|p\rangle_j$:

$$\begin{aligned}|\mathbf{p}\rangle_{123} &\rightarrow e^{iR(X_j)}|\mathbf{p}\rangle_{123}\\ &= (1/\sqrt{2\pi})\int_{-\infty}^{\infty}e^{i\mathbf{p}x+iR(x)}|xxx\rangle_{123}\,dx\,.\end{aligned} \tag{9}$$

This error can be corrected in an analogous way to the errors on single continuous variables: Create redundant states of the nine variables $|\mathbf{p}_{123}\mathbf{p}_{456}\mathbf{p}_{789}\rangle_{1,\ldots,9}$ together with a set of three ancilla variables originally in the state $|000\rangle_{ABC}$, where ancilla variable A is used as the ancilla for the triple of variables 123, B is used for 456, and C is used for 789, then carry out the same error-correcting dynamics as above, but as a function of the continuous variables $\mathbf{p}$ that label the states $|\mathbf{p}\rangle$.

That is, phase errors on the triply redundant state of triply redundant continuous variables can be corrected by applying essentially the same error-correcting routine as before.

To correct any combination of phase and displacement errors on one variable, first apply the $|xxx\rangle$ error-correction routine for error operators of the form $e^{-iQ(P_j)}$ to each of the three triples of variables 123, 456, 789, then apply the $|\mathbf{ppp}\rangle$ error-correction routine for error operators of the form $e^{iR(X_j)}$ to the nine variables as a whole. The basic idea of this continuous quantum error-correcting routine is the same as Shor's binary quantum error-correcting routine [9]: Using triple modular redundancy twice ("triple-triple" modular redundancy) corrects both phase and displacement errors. This sequence of error-correcting steps compensates for the effect of any error operator of the form $e^{-iQ(X_j,P_j)}$, where $Q(X_j,P_j)$ is now a polynomial in the operators X_j, P_j. As any error operator can be approximated arbitrarily closely by composing error operators of this form, the following routine corrects for arbitrary one-variable errors.

To see the error-correction explicitly, use the commutation relation $[X_j,P_j]=i$ to write $e^{-iQ(X_j,P_j)}=\sum_{m,n\geq 0} q_{mn}P_j^m X_j^n$. Look at what happens when an error of this form occurs to one of the variables, for example, the first ($j=1$). We have

$$\begin{aligned}&|\mathbf{p}_{123}\mathbf{p}_{456}\mathbf{p}_{789}\rangle_{1,\ldots,9}|0\ldots 0\rangle_{1',\ldots,9'}|000\rangle_{ABC}\\ &\quad\to\sum_{mn} q_{mn}P_1^m X_1^n(1/\sqrt{2\pi})\int_{-\infty}^{\infty} e^{i\mathbf{p}x}|xxx\rangle_{123}\,dx\\ &\quad\otimes|\mathbf{p}\rangle_{456}|\mathbf{p}\rangle_{789}|0\ldots 0\rangle_{1',\ldots,9'}|000\rangle_{ABC}\,,\end{aligned}\tag{10}$$

which can be rewritten using the decompositions $|x\rangle=(1/\sqrt{2\pi})\int_{-\infty}^{\infty}e^{-ipx}|p\rangle\,dp$, $|p\rangle=(1/\sqrt{2\pi})\int_{-\infty}^{\infty}e^{ipx'}|x'\rangle\,dx'$, as

$$\begin{aligned}&\sum_{mn} q_{mn}(1/\sqrt{2\pi})^3\int_{-\infty}^{\infty}p^m x^n e^{i\mathbf{p}x}e^{-ip(x-x')}\\ &\qquad\times|x'\rangle_1|xx\rangle_{23}\,dx\,dx'\,dp\\ &\quad\otimes|\mathbf{p}\rangle_{456}|\mathbf{p}\rangle_{789}|0\ldots 0\rangle_{1',\ldots,9'}|000\rangle_{ABC}\,.\end{aligned}\tag{11}$$

Now proceed as before, comparing x_i, x_j, x_k, and, if $x_i=x_j=x$, adding $y=x'-x$ to the value of the ancilla state $|0\rangle_{k'}$ and subtracting $x'-x$ from the value of the state $|x'\rangle_k$. Only the first variable and its ancilla state will be affected, resulting in the state

$$\begin{aligned}&\sum_{mn} q_{mn}(1/\sqrt{2\pi})^3\int_{-\infty}^{\infty}p^m x^n e^{i\mathbf{p}x}e^{ipy}|y\rangle_{1'}|xxx\rangle_{123}\,dx\,dy\,dp\\ &\qquad\otimes|\mathbf{p}\rangle_{456}|\mathbf{p}\rangle_{789}|0\ldots 0\rangle_{2',\ldots,9'}|000\rangle_{ABC}\\ &=\sum_{mn} q_{mn}(1/\sqrt{2\pi})\int_{-\infty}^{\infty}x^n e^{i\mathbf{p}x}|xxx\rangle_{123}\,dx\\ &\qquad\otimes|\mathbf{p}\rangle_{456}|\mathbf{p}\rangle_{789}P_{1'}^m|0\rangle_{1'}|0\ldots 0\rangle_{2',\ldots,9'}|000\rangle_{ABC}\,,\end{aligned}\tag{12}$$

where $P_{1'}^m$ acts only as the first ancilla variable. The error-correction routine for states of the form $|xxx\rangle$ has transferred the effect of the P_j^m part of the error operator from the code word to the ancilla.

Similarly, applying the $|\mathbf{ppp}\rangle$ error correction to the state in Eq. (12) transfers the effect of the X_j^n part of the error operator from the code word to the ancilla, resulting in the state

$$\begin{aligned}&(1/\sqrt{2\pi})\int_{-\infty}^{\infty}e^{i\mathbf{p}x}|xxx\rangle_{123}\,dx|\mathbf{p}\rangle_{456}|\mathbf{p}\rangle_{789}\\ &\otimes\left(\sum_{m,n}q_{mn}P_{1'}^m|0\rangle_{1'}X_A^n|0\rangle_A\right)|0\ldots 0\rangle_{2',\ldots,9'}|00\rangle_{BC}\\ &=|\mathbf{p}\rangle_{123}|\mathbf{p}\rangle_{456}|\mathbf{p}\rangle_{789}e^{-iQ(X_A,P_{1'})}|0\ldots 0\rangle_{1',\ldots,9'}|000\rangle_{ABC}\,.\end{aligned}\tag{13}$$

The error has now been corrected. The ancillae can be reset and the procedure repeated to provide ongoing error correction.

In summary, in each term of the polynomial expansion of the error operator, the application of $|xxx\rangle$ error-correcting routine to the triple of continuous variables containing j restores the triple where the error occurred to a superposition of the form $\int_{-\infty}^{\infty}\beta_n(\mathbf{p},\mathbf{p}')\,|\mathbf{p}'\rangle\,d\mathbf{p}'$, where $\beta_n(\mathbf{p},\mathbf{p}')=(1/\sqrt{2\pi})\int_{-\infty}^{\infty}x^n e^{i(\mathbf{p}-\mathbf{p}')x}\,dx$. The subsequent application of the $|\mathbf{ppp}\rangle$ error-correction routine to the triple of triples then restores the nine variables as a whole to the state $|\mathbf{ppp}\rangle_{1,\ldots,9}$. The fact that the state of the ancillae after each error-correcting routine depends only on what errors occurred and not on which code word $|\mathbf{ppp}\rangle_{1,\ldots,9}$ the system was in implies that arbitrary superpositions of the form $\int_{-\infty}^{\infty}\psi(\mathbf{p})\,|\mathbf{ppp}\rangle_{1,\ldots,9}\,d\mathbf{p}$ are also restored by the continuous error-correction routine.

The analog quantum error-correcting routine presented above corrects for errors that are arbitrary polynomials in X_j and P_j and, by extension, to arbitrary single variable errors. It can be enacted in principle using simple operations on the real numbers such as comparing and adding two numbers. What happens when these operations can be performed only to finite precision? By going through the error-correcting routine and following what happens when comparison and addition are performed to finite precision δ, one can verify that the procedure still works as long as (i) the wave function ψ does not vary significantly over scales δ, and (ii) the expectation values for the error operators on the range of ψ do not vary significantly over scales δ. Perhaps the easiest way to see why such inexact error correction still works is to note that, when (i) and (ii) hold for finite precision δ in manipulations of continuous variables, the system behaves like an infinite-dimensional *discrete* system with states $|x_n\rangle = |n\delta\rangle$. The continuous error-correcting scheme above, performed at finite precision, still functions as an error-correcting scheme for the discrete infinite-dimensional system. Similarly, the method described here generalizes in a straightforward fashion to systems that are continuous in one variable and discrete in the complementary variable (e.g., a particle in a box).

We have presented a quantum error-correcting routine for continuous variables. The routine allows the creation of states of a composite system that resist the effects of errors and noise. In practice, of course, performing the continuous "quantum logic gates" necessary to enact the analog error-correcting scheme is likely to prove difficult. For simplicity of exposition, we presented a method for analog quantum error correction based on Shor's original error-correcting routine for qubits. A variety of other continuous quantum error-correcting routines can be constructed based on other discrete quantum codes. In particular, in analogy to [29], it is possible to devise a "perfect" analog quantum error-correcting code using only five continuous variables, although the dynamics of the error correction are more complicated than the simple continuous voting used here [32]. The quantum error-correcting mechanism described here is an example of a feedback loop that preserves quantum coherence as proposed by Lloyd [33]. The nonlinear dynamics cause the ancilla variables to become correlated with the system in a coherent manner, and the information that they possess is used coherently to restore the system to its desired state.

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