Semiclassical Theory of Magnetotransport through a Chaotic Quantum Well

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(Received 8 July 1997)

We develop a quantitative semiclassical formula for the resonant tunneling current through a quantum well in a tilted magnetic field. It is shown that the current depends only on periodic orbits within the quantum well. For example, the theory explains the puzzling evolution of the data near a tilt angle of 30° as arising from an exchange bifurcation of the relevant periodic orbits. [S0031-9007(98)05527-6]

PACS numbers: 73.20.Dx, 05.45.+b, 72.15.Gd

The resonant tunneling diode (RTD) in a magnetic field tilted with respect to the tunneling direction has been extensively studied in recent years as a simple experimental system which manifests the quantum signatures of classical chaos [1–7]. The measured I-V characteristics show resonance peaks which evolve in a complex manner as magnetic field B and tilt angle θ are varied [1,2]. The existence and periodicity of these peaks in the various parameter intervals have been associated with the existence of certain periodic orbits [1] and their bifurcations [2,3]. The link to quantum mechanics has been made by intuitive appeals to Gutzwiller oscillations of the density of states [1], scaling analyses of the exact quantum spectrum [4], and the numerical discovery of sequences of wave functions scarred by periodic orbits [6]. However, previous to this work, it has not been shown that periodic orbits indeed determine the quantum tunneling oscillations in the semiclassical limit. Below we derive a quantitative semiclassical formula for the tunneling current which demonstrates that the current is dominated by periodic orbits and apply the formula to previously unpublished data which reveals an interesting *exchange bifurcation* involving four period-two orbits.

In an RTD under a bias voltage V, tunneling current flows from the emitter state through the double barriers confining the quantum well. The data presented are from an RTD with a 120 nm wide well and experimental details are given in Ref. [2]. When a large magnetic field (>1 T) is applied, the emitter state is quantized into the first few Landau levels. The electric field is normal to the barriers ($\mathbf{E} = E\hat{\mathbf{z}}$), while the magnetic field is tilted in the y-z plane, $\mathbf{B} = \cos\theta\,\hat{\mathbf{z}} + \sin\theta\,\hat{\mathbf{y}}$.

After tunneling into the well through the emitter barrier, the electron will typically begin to lose kinetic energy by optic phonon emission after only 4–5 collisions with the barriers, but will traverse the well hundreds of times before tunneling out. Therefore, the tunneling is sequential and the resonances are substantially broadened by $\hbar/\tau_{\mathrm{opt}}$, where the phonon emission time is $\sim$0.1 ps [1]. For describing this limit the Bardeen tunneling Hamiltonian formalism is appropriate [8,9]. Using this approach, and the approximation that the tunneling rate through the emitter barrier is much less than the rate through the collector barrier (which describes the recent experiments [1,2]), one finds that the current is determined solely by the tunneling rate through the emitter barrier [10] $W_{e\to w}$,

$$j = n_e e W_{e\to w}\,, \tag{1}$$

where n_e is the electron density in the emitter layer.

$W_{e\to w}$ can be calculated from the Fermi's "golden rule" with the coupling given by the square of the matrix element [8,9] between the wave functions Ψ_e and Ψ_w, corresponding to the *isolated* emitter and *isolated* well, respectively. In the limit when the height of the emitter barrier is much larger than the injection energy ε_i, the cyclotron energy and the voltage drop across the barrier, this matrix element can be simplified to

$$M_{nk}^{e\to w} = \frac{\hbar^2}{m}\int d\mathbf{S}\,\Psi_n^e(x,y,0)\,\frac{\partial \Psi_k^w(x,y,0)^*}{\partial z}, \tag{2}$$

where the integration is performed over the inner surface of the emitter barrier $z = 0$.

Because of the translational invariance in the x direction, the classical dynamics within the well can be reduced to 2 degrees of freedom, y, z, [2–5] with an effective potential $V(y,z)$. The well wave functions in Eq. (1) can be reexpressed in terms of the Green function of the isolated well, $G(y_1, z_1 = 0; y_2, z_2 = 0; \varepsilon)$, which is then replaced by its semiclassical approximation [11], determined by *all* classical trajectories connecting the points $(y_1, 0); (y_2, 0) \equiv (y - \Delta y/2, 0); (y + \Delta y/2, 0)$. The emitter state Ψ_e in Eq. (2) involves only the few lowest single-particle levels and can be calculated accurately using a variational approach [12]. $\Psi_e(y)$ is then a linear combination of the lowest few Landau levels, and has spatial extent of order the magnetic length, $l_B \equiv \sqrt{\hbar/eB} \sim \hbar^{1/2}$.

We then obtain for the oscillatory part of $W_{e\to w}$

$$W_{\mathrm{osc}} = \int dp_y \int dy\, f_W(y,p_y) \sum_\gamma \frac{(p_z^\gamma)_i (p_z^\gamma)_f}{(m^*)^2} \int d\Delta y \times \Re\left\{ \frac{8 D_\gamma^{1/2}}{\sqrt{2\pi\hbar i}} \exp\left[-\frac{t_\gamma}{\tau_{\mathrm{opt}}} + i\,\frac{S_\gamma - p_y \Delta y}{\hbar} \right] \right\}, \tag{3}$$

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where $S_\gamma \equiv S_\gamma(y - \Delta y/2, 0; y + \Delta y/2, 0; \varepsilon)$ is the action of the classical trajectory indexed by γ, t_γ is the classical propagation time, D_γ is the appropriate (complex) amplitude [11], and p_i^γ and p_f^γ correspond to the initial and final momenta of the trajectory.

We have also introduced in Eq. (3) the Wigner transform of the emitter wave function, $f_W(y, p_y) = h^{-1} \int d\Delta y \Psi_e(y - \Delta y/2, 0)\Psi_e^*(y + \Delta y/2, 0) \exp(ip_y \times \Delta y/\hbar)$; this function describes the distribution in transverse position and momentum of electrons injected into the well. Since $\Psi_e(y)$ has a width $\sim l_B$, the integrand will be small for $\Delta y > l_B \sim \hbar^{1/2}$. Finally the factor $\exp(-t_\gamma/\tau_{\mathrm{opt}})$ represents the effect of phonon emission.

Consider first the integration over Δy in Eq. (3). Since $\Delta y \sim \hbar^{1/2}$, in the semiclassical limit $\hbar \to 0$, one can expand $S_\gamma(y - \Delta y, y + \Delta y)$ retaining only terms up to second order, and perform exactly the resulting Gaussian integral. Alternatively, we can employ the approach of Berry [13] and perform this integration by stationary phase, which will initially lead to an expression in terms of nonclosed orbits which satisfy the "midpoint rule," and then reexpress this answer to the same accuracy in $\hbar$ using $\Delta y \sim \sqrt{\hbar}$ to arrive at the same result involving only families of closed orbits.

The y integration now involves the rapidly varying phase $\exp[iS_\gamma(y, y)/\hbar]$ for closed orbits beginning and ending at y. Typically these orbits occur in families around a discrete *periodic* orbit μ at which $S_\gamma(y, y)$ is stationary [11]. Since $f_W(y)$ varies on the same spatial scale $\sim\hbar^{1/2}$, we cannot immediately perform the y integral by stationary phase (as is done to derive the trace formula for the total density of states [11]). However, here we can represent all closed orbits to the required accuracy by quadratic expansion of S_γ *around* the discrete set of periodic orbits. One then finds

$$W_{\mathrm{osc}} = \int dy \int dp_y f_W^{(e)}(y, p_y) \sum_\mu \frac{16 p_z^\mu \exp(-T_\mu/\tau_{\mathrm{opt}})}{m^* \sqrt{|m_{11}^\mu + m_{22}^\mu + 2|}} \cos\left[\frac{S_\mu}{\hbar} - \frac{\pi n_\mu}{2} + Q_\mu(\delta y, \delta p_y)\right], \qquad (4)$$

where

$$Q_\mu = \frac{2}{\hbar} \frac{m_{21}^\mu (\delta y)^2 + (m_{22}^\mu - m_{11}^\mu)\delta y \delta p_y - m_{12}^\mu (\delta p_y)^2}{(m_{11}^\mu + m_{22}^\mu + 2)}.$$

Here μ labels the periodic orbit, $\delta y = y - y_\mu$, $\delta p_y = p_y - (p_\mu)_y$, the integer n_μ is the topological index [11] of the periodic orbit, and the 2×2 monodromy matrix [11] (m_{ij}^μ) is calculated at the contact point at the emitter barrier. We have thus shown that the tunneling current depends only on the periodic orbits in the well.

The summation in Eq. (4) is performed over all isolated periodic orbits, both stable and unstable. Near stable islands the motion is regular and we expect semiclassical quantization to yield discrete energy levels and sequences of eigenfunctions localized on the islands [14]. In contrast, near unstable orbits the motion is chaotic and semiclassical theory does not yield discrete levels [11]. This difference can be displayed explicitly by performing exactly the summation over repetitions of the primitive periodic orbits in Eq. (4), yielding

$$W = \frac{8}{m} \sum_\mu (p_\mu)_z \sum_\ell \Delta\left(\frac{T_\mu}{\tau_{\mathrm{eff}}^\mu}, \frac{S_\mu(\varepsilon_\ell)}{\hbar} - \frac{\pi n_\mu}{2}\right) \times \int dy \int dp_y f_W^e(y, p_y) g_\ell^{\mu,\pm}(y, p_y), \qquad (5)$$

where $\Delta(\sigma, \rho) = \sinh(\sigma)/[\cosh(\sigma) - \cos(\rho)]$, and the index $+$ or $-$ denotes stable or unstable orbits. The quantity $\hbar/\tau_{\mathrm{eff}}^\mu$ is an effective level broadening which differs in the two cases.

Note that the function Δ has a peak every time the semiclassical quantization condition $S_\mu(\varepsilon_l) = 2\pi\hbar(n + n_\mu/4)$ is satisfied, and these peaks become delta functions as $\tau_{\mathrm{eff}}^\mu \to \infty$. For stable orbits we find that $\tau_{\mathrm{eff}}^\mu = \tau_{\mathrm{opt}}$, so if we neglect phonon scattering ($\tau_{\mathrm{opt}} \to \infty$) we do recover perfectly discrete contributions to the tunneling current. In the stable case the argument of S_μ in Eq. (5) is $\varepsilon_\ell^+ \equiv \varepsilon - \hbar\omega_\perp^{\mu,+}(\ell + 1/2)$ which may be interpreted as the energy of longitudinal motion along the orbit. Because of the harmonic approximation the quantization of the transverse oscillations around the periodic orbit simply yields equally spaced levels [14] with spacing $\hbar\omega_\perp^{\mu,+}$, where the frequency $\omega_\perp \equiv \phi_\mu/T_\mu$, ϕ_μ is the winding number [11], and T_μ the period of the orbit. So the discrete energies at which tunneling occurs *are* the correct semiclassical energy levels of the well.

The amplitude of each contribution is given by the coefficient functions g_ℓ in Eq. (5), which are the Wigner transforms of the harmonic oscillator wave functions corresponding to these transverse modes

$$g_\ell^{\mu,+}(y, p_y) = (-1)^\ell L_\ell(2|\tilde{Q}_\mu|) \exp(-|\tilde{Q}_\mu|), \qquad (6)$$

where L_ℓ is the Laguerre polynomial and $\tilde{Q}_\mu = |2 + \mathrm{Tr}[M]|^{1/2}|2 - \mathrm{Tr}[M]|^{-1/2} Q_\mu$.

Since the result (6) is based on the harmonic approximation within a stable island, we may include only modes up to $\ell_{\max}$, which is given by the ratio of the island area to $\hbar$. Phonon scattering smears out each of these discrete contributions to W_{osc} over an energy range $\hbar/\tau_{\mathrm{opt}}$.

In contrast, for unstable periodic orbits we find that $\tau_{\mathrm{eff}}^\mu = \tau_{\mathrm{opt}}/(1 + \ell\lambda_\mu\tau_{\mathrm{opt}})$, where λ_μ is the Lyapunov exponent near the orbit μ. Hence this time is finite and equal to $1/\lambda_\mu$ in the absence of phonon scattering. Therefore, instability acts as a sort of intrinsic level broadening, and each periodic orbit (PO) describes a contribution due to a cluster of levels. The peak of the function Δ in Eq. (5) corresponding to the mean energy of the cluster is given by $S_\mu(\varepsilon) = 2\pi\hbar(n + n_\mu/4)$. The weight functions

$$g_\ell^{\mu,-}(y, p_y) = (-1)^\ell \Re\left\{L_\ell(2i\tilde{Q}_\mu)\exp(i\tilde{Q}_\mu) \times \left(1 + i\,\frac{\sin(S_\mu/\hbar - \pi n_\mu/2)}{\sinh(T_\mu/\tau_{\mathrm{eff}}^\mu)}\right)\right\} \quad (7)$$

are related to Wigner functions *averaged* [13] over the eigenstates of the cluster. For each unstable PO the high-ℓ contributions are exponentially damped, and the main contribution to the tunneling rate is given by the $\ell = 0$ term.

We now have a rigorous criterion for which periodic orbits contribute substantially to the tunneling current in Eq. (5). The injection function $f_W(y, p_y)$ is centered on y_i and $p_y = 0$ with widths $\sim l_B, \hbar/l_B$ in y, p_y. Weight functions g_μ are centered at $y_\mu, (p_y)_\mu$ with widths $\sim l_\mu = (2\hbar|m_{12}^\mu|)^{1/2}|4 - \mathrm{Tr}^2[M_\mu]|^{-1/4}$ and $\hbar/l_\mu$, respectively. When the real and momentum space peaks of these two functions overlap the PO is semiclassically "accessible" and makes a substantial contribution to the tunneling current. This criterion is illustrated for relevant period-two orbits in Fig. 1. Given the relevant *classical* information for any periodic orbit reaching the emitter, its contribution to the tunneling current can be calculated from Eqs. (5)–(7) [15]. We now apply this formulation to understand aspects of the experimentally observed I-V characteristics.

In the recent periodic orbit theory [5] it was shown that within the set of POs which collide with the collector n times (period-n orbits), there exist orbits which collide with the emitter m times, where $m \leq n$, and it is useful to classify POs by the two integers (m, n). At $\theta = 0$ the only resonances observed in the I-V characteristic are associated with Bohr-Sommerfeld quantization of the $(1, 1)$ orbit which traverses the well with zero cyclotron energy. When $\theta \neq 0$, additional resonances appear corresponding to doubling or tripling of the frequency of peaks [2]. These new peaks are associated with the existence of period-two and period-three orbits which appear and disappear as a result of bifurcations [5]. Here we focus on the peak doubling in the interval $29° < \theta < 34°$, where there are four most relevant orbits [5], denoted by $(1, 2)_1$, $(1, 2)_2$, $(1, 2)_1^*$, $(1, 2)_2^*$.

The evolution of these orbits with magnetic field or voltage is represented by the four colored lines in Fig. 1. We recall [3–5] that under experimental conditions the classical mechanics depends only on two parameters $\beta = c_0 B/\sqrt{V}$ [where $c_0 \approx 3.1(e/m^*)^{1/2}d$ and $d = 120$ nm is the width of the well] and the tilt angle θ. As β increases from zero for fixed θ, these four $(1, 2)$ orbits appear in bifurcations [5,7] and then disappear pairwise at higher β in the inverse tangent bifurcations already mentioned. These orbits can be specified by the coordinate y_μ of their one collision with the emitter (which is $\propto v_x$; see Fig. 1). The most relevant orbits here are the $(1, 2)_2$ and $(1, 2)_1^*$. The semiclassical width l_μ of these orbits around y_μ is denoted by the gray-scale regions (calculated for $B = 8$ T). The width of the injection function $f_W(y)$ is denoted by the hatched regions. Whenever these regions overlap for some value of β the orbit is accessible and Eq. (5) predicts that a peak-doubling region will appear in the B-V parameter space along the parabola corresponding to that value of β (see Fig. 2).

Figures 1(a)–1(c) depict a fascinating feature of the classical dynamics, noted in Ref. [5], which occurs near $\theta = 31°$ for the parameters of Ref. [2]. At $\theta = 29°$ the accessibility intervals for the $(1, 2)_2$ orbit and the $(1, 2)_1^*$ cover the entire interval $4.3 < \beta < 10.9$ overlapping briefly around $\beta = 7.5$. Thus, one expects a large region of peak doubling in the B-V plane with no gaps as observed [2]. However, we can now see that the low voltage and high voltage oscillations are due to these two *different* orbits, and we expect an abrupt amplitude change around $\beta \approx 7.3$ ($V \approx 0.44$ V for $B = 8$ T). This is observed clearly in the data of Fig. 2(a), where the amplitude change is compared to theory. The theory is found to predict the ratio of the amplitudes between the low and high voltage regions (above 0.45 V) with only 25% mean error.

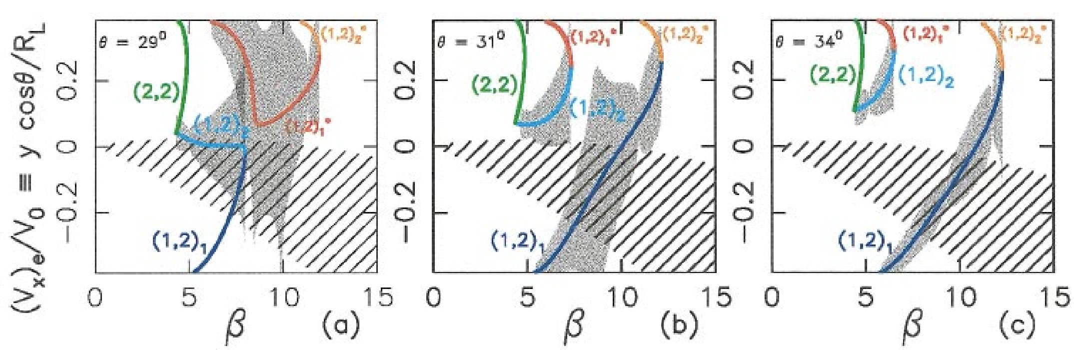

FIG. 1(color). Bifurcation diagrams for the four period-two orbits relevant to the peak doubling in the interval $\theta = 29°$–$34°$ (see text). Colored lines indicate y coordinate of the single collision with the emitter; these lines coincide at bifurcations. $\theta = 29°$ (a) indicates behavior before the exchange bifurcation; $\theta = 31°$ (b) and 34° (c) after. Shading represents the localization lengths l_μ associated with the relevant orbits [$(1, 2)_2$ and $(1, 2)_1^*$ for (a), and $(1, 2)_2$ and $(1, 2)_1$ for (b),(c)]. Hatched region denotes semiclassical width of emitter state; overlap indicates a large contribution to the tunneling current.

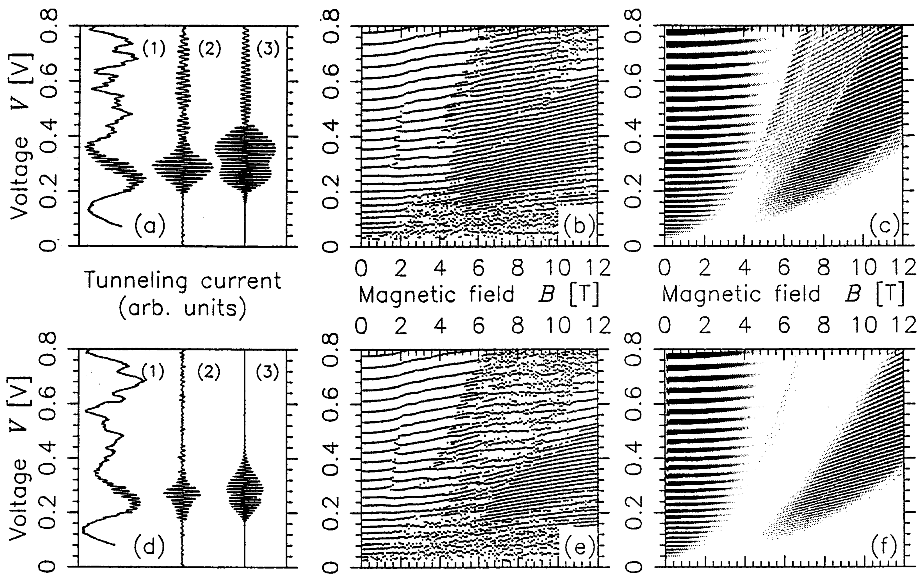

FIG. 2. (a),(d): Resonant tunneling I-V traces for $\theta = 31°, \theta = 34°$ at $B = 8$ T. Trace (1) is raw experimental data, trace (2) is same data, filtered to retain only period-two oscillations, trace (3) is semiclassical prediction from Eq. (5). The modest discrepancies in the shape of the envelope of the amplitude of the oscillations is due to the inaccuracy of the quadratic semiclassical theory near the bifurcation which occurs around 0.3 V at 8 T. (b),(e): Peak positions vs voltage and magnetic field, determined from multiple sets of experimental I-V data at $\theta = 31°, 34°$. (c),(f): Semiclassical I-V oscillations for same, note gray scale indicates relative amplitudes, not just peak positions. Note disappearance of high voltage oscillations at $\theta = 34°$ due to movement of $(1,2)_2$ orbit away from accessibility after the exchange bifurcation [see Figs. 1(b), 1(c), and text].

At $\theta \approx 31°$ the four orbits undergo an exchange bifurcation [16] so that the $(1,2)_1$ is paired with $(1,2)_2^*$, whereas the $(1,2)_1^*$ orbit is now paired with the $(1,2)_2$ [see Fig. 1(b)]. As θ is slightly increased to 34° the $(1,2)_1^*$-$(1,2)_2$ pair which gives rise to the high voltage oscillation moves away from semiclassical accessibility with the emitter [see Fig. 1(c)]. At the same time the $(1,2)_1$ orbit replaces the $(1,2)_1^*$ orbit and is still highly accessible at low voltages (high β). Thus we expect the high voltage oscillations to disappear at $\theta = 34°$ while the low voltage oscillations persist. This is seen clearly in Fig. 2(d), again in good qualitative and quantitative agreement with theory. In Figs. 2(b) and 2(e) the peak position data are plotted in the entire B-V plane against the semiclassical prediction based on the contributions of these periodic orbits. Again good agreement is found [17] demonstrating the possibility of a quantitative semiclassical description.

We acknowledge helpful conversations with Gregor Hackenbroich, Tania Monteiro, and John Delos. The work of A. D. S. and E. N. was supported by NSF Grant No. DMR-9215065.

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