Faraday Instability in a Multimode Laser

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We show theoretically and experimentally that a basic pattern formation mechanism in hydrodynamics, the Faraday instability, can be observed in optics (for modulated class B lasers with inhomogeneous broadening). As for the Faraday problem, stationary waves are excited parametrically and wave numbers are selected by the dispersion curve of the medium. The mechanism is evidenced by a multiple-scale analysis of the physical laser model, and is confirmed experimentally using a Nd-doped fiber laser. [S0031-9007(98)05957-2]

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Pattern formation has been the subject of intensive investigations during the past decade. A number of universal mechanisms for morphogenesis have been identified, which has led to distinct criteria for wavelength selection as a result of a primary instability. This selection can be determined by geometrical constraints as in fluid convection, or by intrinsic properties as in the Turing instability [1]. A third mechanism is the excitation of waves by an external *spatially uniform* modulation. This is the case of the Faraday instability, known since the 19th century [2]. In an open container of fluid, modulation of the vertical position at a frequency ω_m can typically induce a wave at the subharmonic $\omega_m/2$, resulting from a parametric instability. The wave number k_c of this wave is related directly to the modulation frequency through the linear dispersion relation of the medium: $\omega_m/2 = f(k_c)$ [1,3,4]. Studies on this subject from the point of view of nonlinear dynamics have led to an impressive set of new observations [5], in particular the evidence of spatiotemporal chaos [6,7], spatiotemporal intermittency [8], and quasicrystalline waves [9]. However, the generality of these results exceeds the domain of surface waves dynamics as shown explicitly by Coullet *et al.* [10]. These phenomena belong to the more general class of *dispersion-induced patterns*. The conditions for such instabilities to occur are the following: (i) The system must be a (discrete or continuous) propagation medium (for example, a chain of coupled oscillators) and (ii) the external modulation must affect the system *uniformly*. The last condition allows a parametric excitation of the waves. Although these conditions seem *a priori* not too severe, Faraday-like instabilities have rarely been observed in areas different from hydrodynamics, except for spin waves [11,12], and crystallization dynamics [13].

In this Letter, we report the observation of dispersion-induced patterns in the optical spectrum of a laser. An inhomogeneously broadened class-B laser can behave as a chain of coupled oscillators, each being associated to one longitudinal mode [14]. It can therefore be considered as a spatiotemporal system for which the (discrete) spatial variable is the mode index j, and the information that propagates is the mode intensity. Thanks to the local coupling, linear damped waves can propagate in this set of modes (and thus in the optical spectrum). Since the dispersion relation of the waves is known [14], and global modulation of the laser is easily achieved by modulating the pump, this laser is a good candidate for the observation of dispersion-induced ("Faraday-like") instabilities.

Our study of the problem will be the following. From the physical model of the laser [14–16], we first determine long-time amplitude equations using a multiple-scale analysis. We then analyze the primary instability. The properties of the bifurcation are compared to the known properties of the Faraday experiment (i.e., wave-number selection, temporal period and stationary wave nature). Finally, we verify our analytical predictions numerically and experimentally, using a Nd-doped fiber laser subjected to pump modulation.

We consider an inhomogeneously broadened class-B laser without phase-sensitive interactions. The state of such a laser can be described by a set of mode intensities $s_j(t)$, and a continuous set of population inversion $d(\xi, t)$ [15–17]. In dimensionless form, the model reads

$$\partial_t s_j = -s_j + s_j \int_{-\infty}^{+\infty} \beta(\xi_j - \xi) d(\xi) d\xi \,, \tag{1a}$$

$$\partial_t d = \gamma \left[g(\xi) d^0(t) - \left(1 + \sum_l \beta(\xi - \xi_l) s_l \right) d \right]. \tag{1b}$$

In these equations, the emission optical frequencies associated with each population class ξ, and to each mode ξ_j, play the role of "spatial" variables (not time variables). The intensities and populations evolve with a time

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scale much slower than the optical ones: The time t is measured in units of the cavity lifetime τ_c, typically in the microsecond range. The parameter γ is the population inversion rate normalized by the photon decay rate $1/\tau_c$ ($\gamma \ll 1$). $d^0(t) = A[1 + m\cos(\omega_m t)]$ denotes the modulated pumping rate where A, m, and ω_m represent its average, amplitude, and frequency, respectively. ω_m and m are our main control parameters, and ω_m is chosen close to the relaxation frequency $\omega_R = \sqrt{\gamma(A - 1)}$ [14] (typically in the kHz-MHz range).

The cross-saturation coupling coefficients are defined by $\beta(\xi) = [1 + (\frac{\xi}{\Delta})^2]^{-1}/\pi\Delta$, where Δ is the homogeneous width. The Lorentzian shape of $\beta(\xi)$ ensures a *local* coupling with an interaction range Δ. Nonuniformities of the laser spectrum are modeled by the selective pumping $g(\xi)$. In the theoretical work, we will consider an infinite medium with $g(\xi) = 1$. In the numerical simulations, $g(\xi)$ will be taken Gaussian: $g(\xi) = e^{-(\xi-\xi_0)^2/2\sigma^2}$, with $\sigma = 30\Delta$.

We propose a bifurcation analysis of Eq. (1), using the modulation amplitude as the control parameter, and for ω_m close to ω_R. Considerable simplification of the mathematical problem is achieved by taking the continuous limit $\Delta^{-1} \to 0$ for the mode intensities s_j. This limit is motivated by the large number of modes per homogeneous width Δ, observed in our experiments discussed below. We thus approximate the sum in (1) by a Riemann integral. We next consider the case $g(\xi) = 1$, and introduce the following useful change of variables [18]: $s = (A - 1)(1 + X)$, $d = 1 + \omega_R Y$, and $t = \tau/\omega_R$. We obtain

$$\partial_\tau X = (1 + X)\int_{-\infty}^{+\infty} \beta(\xi - \xi')Y(\xi')d\xi', \tag{2a}$$

$$\partial_\tau Y = -\left(1 + \frac{2\epsilon}{F}Y\right)\int_{-\infty}^{+\infty} \beta(\xi - \xi')X(\xi')d\xi' - 2\epsilon Y + \delta\cos\Omega_m\tau, \tag{2b}$$

where $F = A/(A - 1)$. We then note that the laser parameters are grouped into three parameters ϵ, δ, and Ω_m defined as

$$\epsilon = \frac{A}{2}\sqrt{\frac{\gamma}{A - 1}}, \quad \delta = \frac{mA}{A - 1}, \quad \text{and} \quad \Omega_m = \frac{\omega_m}{\omega_R}. \tag{3}$$

The first parameter is small and measures the damping of the laser-free relaxation oscillations in units of ω_R. The second and third parameters are the normalized amplitude and frequency of the modulation.

The small dissipation coefficient ϵ and the fact that $\Omega_m \approx 1$ suggest that one should apply a multiple-scale perturbation analysis [19]. After introducing $T = \Omega_m\tau$, slow time and space variables $\theta = \epsilon\tau$, $\zeta = \epsilon\Delta^{-1}\xi$, we seek a solution of Eq. (2) of the form,

$$X(T,\theta,\xi,\zeta,\epsilon) = \epsilon X_1(T,\theta,\xi,\zeta) + \ldots, \tag{4a}$$

$$Y(T,\theta,\xi,\zeta,\epsilon) = \epsilon Y_1(T,\theta,\xi,\zeta) + \ldots. \tag{4b}$$

We assume $\Omega_m \approx 1$ and δ small, and introduce the following expansions of the parameters: $\Omega_m = 1 + \sigma_1\epsilon + \ldots$ and $\delta = 16\epsilon^2\delta_2 + \ldots$. After substituting these expansions into Eq. (2), we obtain a sequence of linear problems to solve. The leading order problem admits the solution,

$$X_1(\xi) = 4i[Ue^{iT} + \mathrm{R}e^{i(T/2-k_0\xi)} + \mathrm{L}e^{i(T/2+k_0\xi)}] + \text{c.c.}, \tag{5a}$$

$$Y_1(\xi) = -4[Ue^{iT} + \mathrm{R}e^{i(T/2-k_0\xi)} + \mathrm{L}e^{i(T/2+k_0\xi)}] + \text{c.c.}, \tag{5b}$$

where the amplitudes of the uniform component U, and of the waves R and L, depend on the slow variables θ and ζ. $(\Omega_m/2, k_0)$ satisfies the dispersion relation [14],

$$\Omega_m/2 = e^{-|k_0|\Delta}. \tag{6}$$

The unknown amplitudes U, R, and L are obtained from higher order solvability conditions, and satisfy the following amplitude equations:

$$\partial_\theta R - \tfrac{1}{2}\partial_\zeta R = -R + UL^*, \tag{7a}$$

$$\partial_\theta L + \tfrac{1}{2}\partial_\zeta L = -L + UR^*, \tag{7b}$$

$$\partial_\theta U + i\sigma_1 U = -U - 2RL - \delta_2. \tag{7c}$$

As for the Faraday instability, we note from Eqs. (7) that the amplitudes R and L of the two waves (with group velocities $-1/2$ and $+1/2$, respectively) are *parametrically excited* by the uniform amplitude U [i.e., $R = L = 0$ is always a solution of Eqs. (7)]. Moreover, it is worth noticing that Eqs. (7) are similar to equations for the optical parametric oscillator [20,21].

From an analysis of the solutions of Eq. (7), we find a bifurcation to a stationary wave that satisfies all Faraday conditions. Indeed, we note that (*i*) the uniform state ($R = L = 0$) becomes unstable at a period-doubling bifurcation point, (*ii*) the bifurcation leads to a stable stationary wave ($|R| = |L|$), and (*iii*) the selected wave number is determined by the dispersion relation (6). Indeed, introducing $R = ae^{i\delta K\zeta}$ and $L = R^*e^{i\phi}$ into Eq. (7) leads to the following expression of the critical modulation amplitude:

$$m_c(\delta K) = \tfrac{16}{F}\epsilon^2\sqrt{1 + \delta K^2}\sqrt{1 + (\Omega_m - 1)^2/\epsilon^2} + O(\epsilon^3). \tag{8}$$

From (8), we note that the first instability of the uniform solution occurs as $\delta K = 0$. This means that the first instability corresponds to a critical wave number $k_c = k_0$, which satisfies the dispersion relation (6).

The experimental setup used for verifying these predictions is based on the laser described in Ref. [14]. It consists basically of a multimode Nd-doped fiber laser cooled at 77 K, with an emission width 50 Å, much larger than the homogeneous width $\Delta \approx 0.8$ Å. Concerning the cavity, the only technical difference with Ref. [14] lies

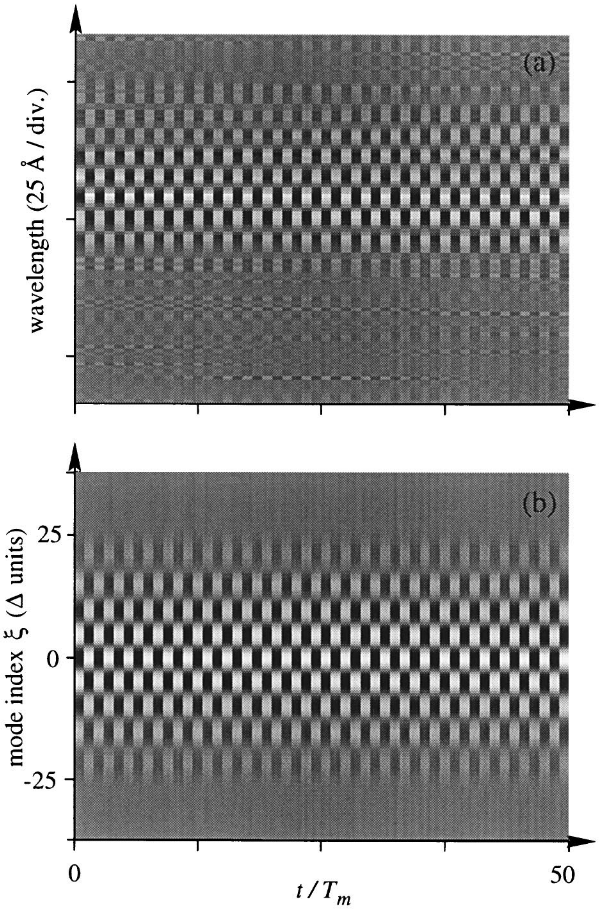

FIG. 1. Evolution of the mode intensities just above the instability threshold ($m = 0.12$, $\omega_m = \omega_R$, and $A = 2.3$). T_m is the modulation period. (a) Experiment ($\omega_R = 62$ kHz); (b) numerical integration of Eqs. (1) with $\epsilon = 0.0365$ and $\sigma = 30\Delta$. One optical spectrum is sampled at each modulation period. The average spectrum has been subtracted from the spectrochronogram, and white corresponds to high intensities.

in the reflection coefficient of the output coupler (80% here). However, the laser is now investigated in utterly different conditions from [14] (where only the linear response was considered). Here, we concentrate on the nonlinear response to a sinusoidal modulation of pump power. For this purpose, we modulate sinusoidally the current injected into the diode laser. The mode evolution is monitored on a silicium CCD array, and one optical spectrum is recorded at each modulation period.

Pattern formation appears above a critical threshold, and satisfies the Faraday instability scenario. Just above threshold, this pattern consists of a stationary wave oscillating with frequency $\omega_m/2$ [Fig. 1(a)]. In order to determine whether it is actually a dispersion-induced pattern, we examine its Fourier transform in both "space" and time (Fig. 2). We observe two peaks, located at $(\pm k_c, \omega_m/2)$, which satisfies the dispersion relation (6) or, equivalently, $\omega(k) = \omega_R \exp(-|k|\Delta)$ [14]. These results are well reproduced by numerical integration of

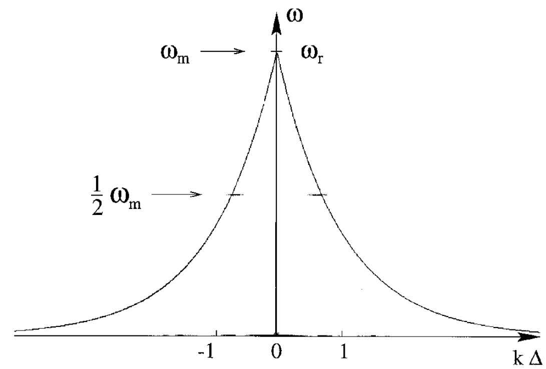

FIG. 2. Fourier transform in both space and time of the experimental regime shown in Fig. 1(a). The Nyquist temporal frequency is $\omega_m/2$, and black is associated with high power spectral densities. The linear dispersion curve [$\omega(k) = \omega_R \exp(-|k|\Delta)$, with $\Delta = 0.8$ Å] is superposed. Note that ω is NOT an optical frequency (see text).

the model equations (1), for the same values of the parameters. We have used a pseudospectral method [22] and, because $\Delta \approx 2000 \gg 1$, we have approximated the discrete sum by an integral. The pattern displayed in Fig. 1(b) clearly exhibits the same stationary wave, with frequency $\omega_m/2$. Moreover, the Fourier transform reveals the same agreement with the dispersion relation (6).

We also examined the quantitative agreement between the theoretical predictions, and the experimental values of the bifurcation point. Figure 3 represents the experimental dependence of the threshold modulation amplitude m_c, with respect to the frequency ω_m, together with the least-squares fit of the theoretical function (8). This fit yields to the values of $\omega_R/2\pi = 129$ kHz and $\epsilon = 0.048$, which compare well with their actual values of 132 kHz and 0.047, respectively [23]. We also note that qualitative and quantitative agreements remain good even in regions where the near-resonance and low-amplitude conditions are poorly satisfied.

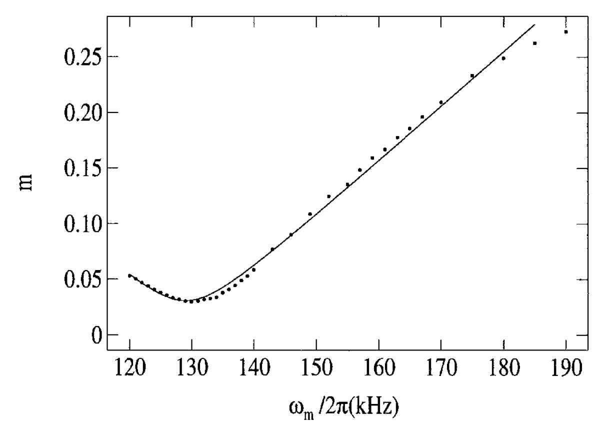

FIG. 3. Instability threshold (critical modulation amplitude) versus modulation frequency. Dots: experimental results ($A = 5.5$); solid line: fit of the theoretical result $m_c(0)$ given by Eq. (8).

In conclusion, modulation of an inhomogeneously broadened multimode laser can excite dispersion-induced patterns in its spectrum of emission. We call this bifurcation a Faraday instability, because it verifies the following three points: (*a*) The bifurcation leads to stationary waves, (*b*) wavelength selection is related to the modulation frequency through the linear dispersion curve of the medium, and (*c*) the bifurcation is the result of a parametric instability, meaning that the uniform state ($k = 0$) is still a solution of the problem. The perspectives of our paper are twofold. Investigations of secondary instabilities observed experimentally for higher modulation amplitudes will tell us how generic is the Faraday mechanism. In particular, we are interested in determining the effects of *finite range* coupling [24] (one mode is not only coupled to its nearest neighbors but also to others within a range Δ). A second perspective of our paper is to understand the effects of space dependence of the control parameters in dispersion-induced instabilities (for example, the effect of inhomogeneous pumping) which are always present in the experiments.

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