Dynamical Properties of the Pinned Wigner Crystal

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We study various dynamical properties of the weakly pinned Wigner crystal in a high magnetic field. Using a Gaussian variational method, we can compute the full frequency and field dependence of the real and imaginary parts of the diagonal and Hall conductivities. The zero temperature Hall resistivity is independent of frequency and remains unaffected by disorder at its classical value. We show that, depending on the inherent length scales of the system, the pinning peak and the threshold electric field exhibit strikingly different magnetic field dependences. [S0031-9007(98)05864-5]

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It is known that the experimentally elusive Wigner crystal (WC), predicted to form in low density two-dimensional electron gases [1], can be induced in denser systems by subjecting them to strong perpendicular magnetic fields. The quest for the Wigner crystal in monolayer [2–5] and bilayer [6] quantum Hall samples indicated the existence of a quaint insulating state at filling fractions where crystallization was theoretically expected. Detailed studies [2–7] of the insulating state revealed that the diagonal resistivity ρ_{xx} diverges as the temperature $T \to 0$ and shows activated behavior at finite T, whereas the Hall resistivity ρ_{xy} is temperature independent with a value close to the classical one.

Although transport and linear dc conductivity and the luminescence spectrum of radiative recombination measurements [8] were consistent with interpretations in terms of a pinned WC, the finite value for ρ_{xy} was unexpected. This prompted other interpretations of the observed insulating phase [9,10] and, in particular, the existence of a new phase, the Hall insulator (HI) [10]. The HI is defined as a phase where $\lim_{\omega\to 0} \frac{\sigma_{xy}(\omega)}{\omega^2} = \text{const}$, which yields $\rho_{xx} \to \infty$ and a finite ρ_{xy} in the limit $T \to 0, \omega \to 0$. This was proved only for *noninteracting electrons* in a random potential (i.e., an Anderson insulator with a magnetic field), and qualitative arguments suggested that it holds for interacting systems as well. However, none of these arguments take into account the possible local crystalline order which could result in radically different physics as compared to the disordered electron fluid. Indeed, periodicity plays an important role in other disordered systems, such as vortex lattices [11].

Direct observation of local crystalline order being difficult, it is thus of prime importance to investigate in detail the transport properties of a pinned WC. One of the few theoretical attempts made to predict these properties was [12] in the related problem of charge density waves (CDW) in a magnetic field. The harmonic approximation used, however, did not allow the extraction of the detailed frequency dependence of the conductivities. Later studies [13] focused on the sliding state and the effects of free carriers, or on the effect of strong disorder [14]. A noteworthy point is that none of these calculations consider both a lattice structure and modulation of disorder at scales smaller than the lattice spacing [11]. This feature which is absent in CDW turns out to play a crucial role for the WC.

In this Letter, we compute for the first time the dynamical conductivities of a weakly pinned WC. We find that even some of the main features derived in Ref. [12] are incorrect. The results we obtain provide a basis for comparison with recent experiments which map out the low frequency behavior of the conductivity [15,16].

Our starting point is the WC in a magnetic field B with lattice spacing a modeled by an elastic Hamiltonian [12]. The electrons at site i are displaced from their mean equilibrium positions $\mathbf{R}_i$ by $\mathbf{u}(\mathbf{R}_i, t)$. We also take into account the Coulomb repulsion between density fluctuations. We use the following decomposition: $\mathbf{u}(\mathbf{q}) = \hat{\mathbf{q}} u_L(\mathbf{q}) + \hat{\mathbf{q}} \wedge \hat{\mathbf{z}} u_T(\mathbf{q})$, where L, T denote the longitudinal and transverse components. The corresponding action in the imaginary time formalism is

$$\begin{aligned} S[u] = \int_{\mathbf{q}} \sum_n & [u^L_{q,\omega_n}(\rho_m \omega_n^2 + cq^2 + dq) u^L_{-q,-\omega_n}] \\ & + u^T_{q,\omega_n}(\rho_m \omega_n^2 + cq^2) u^T_{-q,-\omega_n} \\ & + \rho_m \omega_c \omega_n [u^L_{q,\omega_n} u^T_{-q,-\omega_n} - u^L_{-q,-\omega_n} u^T_{q,\omega_n}] \\ & + \int_{\mathbf{x}} \int_0^{\beta\hbar} d\tau W(x) \rho(x,\tau), \end{aligned} \tag{1}$$

where ρ_m, ρ_c are the mass and charge densities. c and d are the shear and bulk modulus respectively. For the WC, Coulomb forces give [13,17] a bulk modulus $d = \rho_c^2/\epsilon_0$ much greater than the shear modulus $c = \rho_c^2 a/\epsilon_0$ (ϵ_0 is the dielectric constant of the substrate). $\omega_c = \rho_c B/\rho_m$ is the cyclotron frequency and the Matsubara frequencies at temperature T are $\omega_n = 2\pi n/\beta\hbar$, where $\beta = 1/T$. Equation (1) shows that Coulomb interactions affect only the longitudinal modes. B couples the transverse and longitudinal modes. $\langle\rangle$ denote averages over quantum and thermal fluctuations and $\overline{()}$ are disorder averages. For the pure system the quantum fluctuations result in $\langle \mathbf{u}^2 \rangle \sim l_c^2$, where $l_c = \sqrt{\hbar/eB}$ is the magnetic length.

The disorder potential $W(\mathbf{x})$ is assumed Gaussian, a good approximation for weak disorder [18], and of a range of correlation r_f: $\overline{W(\mathbf{x})W(\mathbf{x}')} = \Delta\delta_{r_f}(\mathbf{x} - \mathbf{x}')$. Since the precise microscopic nature of the disorder is hard to ascertain, we retain both Δ and r_f as parameters. Disorder couples to the density of electrons $\rho(\mathbf{x}, t) = \sum_i \delta(\mathbf{x} - \mathbf{R}_i - \mathbf{u}(\mathbf{R}_i, t))$. Using the decomposition of the density into lattice harmonics [11] (valid in the absence of topological defects) and replicas to average over disorder, we obtain the effective action (up to terms leaving the conductivity unaltered)

$$S_{\text{eff}} = \sum_a H[u^a] - \frac{1}{2\hbar}\int_{\mathbf{x}}\int\int d\tau d\tau' \times \sum_{a,b,\mathbf{K}} \Delta_K \cos\{\mathbf{K}\cdot[\mathbf{u}^a(\mathbf{x}\tau) - \mathbf{u}^b(\mathbf{x},\tau')]\}, \quad (2)$$

where a, b denote the replica indices, $\mathbf{K}$ are the reciprocal lattice vectors, and $\Delta_K \sim \Delta\exp(-\mathbf{K}^2 r_f^2)$. Note that it is important to retain all harmonics. Actions similar to (2) can be used to describe 3D classical problems such as vortex lattices with correlated disorder [18] and long range interactions.

To study this model we use the Gaussian variational method (GVM). Unlike previously used methods [12], the GVM is self-consistent, has no undetermined adjustable parameters, and also incorporates important physical features of the problem such as the existence of many metastable states. It thus allows one to go beyond simple static arguments. We introduce the variational action [18]

$$S_0 = \frac{1}{2}\int_{\mathbf{q}}\sum_n u^a_{\alpha,q,\omega_n} G^{ab-1}_{\alpha\beta}(q,\omega_n) u^b_{\beta,-q,-\omega_n}, \quad (3)$$

where the Green functions $G^{ab}_{\alpha\beta}$ are the variational parameters ($\alpha, \beta = L, T$ and summations over repeated indices are implicit). They are determined by solving the self-consistent saddle point equations obtained by extremizing the variational free energy $F_{\text{var}} = F_0 + \langle S_{\text{eff}} - S_0\rangle_0$. The method extends the one used in [18], and all technical details will be presented in [19]. As in [18] the solution has a replica symmetry broken structure necessary to correctly describe the localization. The final result [19] is a closed set of equations for the connected part of the Green function $G^{-1}_{c\alpha\beta} = \sum_b G^{-1}_{\alpha\beta,ab}$ which determines all physical quantities of interest here. These equations are

$$\begin{aligned} G^{-1}_{cT} &= (cq^2 + \rho_m\omega_n^2) + F + \frac{\rho_m^2\omega_n^2\omega_c^2}{(cq^2 + dq + \rho_m\omega_n^2 + F)}, \\ G^{-1}_{cL} &= (cq^2 + dq + \rho_m\omega_n^2) + F + \frac{\rho_m^2\omega_n^2\omega_c^2}{(cq^2 + \rho_m\omega_n^2 + F)}, \\ G^{-1}_{cLT} &= \rho_m\omega_n\omega_c + \frac{(cq^2 + \rho_m\omega_n^2 + F)(cq^2 + dq + \rho_m\omega_n^2 + F)}{\rho_m\omega_n\omega_c}, \end{aligned} \quad (4)$$

with $F = I(\omega_n) + \Sigma(1 - \delta_{n,0})$. The localized phase is characterized by a nonzero Σ, from which a length scale l can be defined through $\Sigma = cl^{-2}$. $I(\omega_n)$ is defined as

$$I(\omega_n) = \frac{2}{\hbar}\int_0^\beta d\tau[1 - \cos(\omega_n\tau)][V'(\tilde{B}(\tau)) - V'(B)],$$

where $\tilde{B}(\tau) = \frac{1}{2}\overline{\langle(u(0,\tau) - u(0,0))^2\rangle} = \frac{1}{2}[\tilde{B}_L(\tau) + \tilde{B}_T(\tau)]$ the local diagonal correlation and the off-diagonal part B are given by

$$\begin{aligned} \tilde{B}_{T,L}(\tau) &= \frac{2\hbar}{\beta}\int_q\sum_n G_{cT,L}[1 - \cos(\omega_n\tau)], \\ B &= \frac{\hbar}{\beta}\int_{\mathbf{q}}\sum_{n\neq 0, i=T,L} G_{c,i}(q,\omega_n) + \frac{1}{cq^2 + dq + \Sigma} + \frac{1}{cq^2 + \Sigma}. \end{aligned} \quad (5)$$

Finally the equations close as Σ is itself determined by

$$1 = -2V''(B)\int_{\mathbf{q}}\frac{1}{(cq^2 + \Sigma)^2} + \frac{1}{(cq^2 + dq + \Sigma)^2}.$$

The primes denote derivatives. All information on the disorder is contained in the auxiliary function $V[B] = (4\hbar)^{-1}\sum_K \Delta_K \exp(-K^2 B)$.

In this paper, we focus on the transport properties, but other quantities such as positional correlation functions can also be computed [18,19]. The dynamical conductivities are given by the standard analytical continuation $\sigma_{\alpha\beta}(\omega) = i\rho_c^2\omega G_{\alpha\beta}(q = 0, \omega + i\epsilon)$. Rotational invariances, parity and time reversal breaking by B, imply

$$\begin{aligned} \sigma_{xx} &= \sigma_{yy} = \rho_c^2\frac{i\omega[-\rho_m\omega^2 + \Sigma + I(\omega)]}{[\Sigma - \rho_m\omega^2 + I(\omega)]^2 - \rho_m^2\omega^2\omega_c^2}, \\ \sigma_{xy} &= -\sigma_{yx} = \rho_c^2\frac{i\omega[-i\rho_m\omega\omega_c]}{[\Sigma_1 - \rho_m\omega^2 + I(\omega)]^2 - \rho_m^2\omega^2\omega_c^2}. \end{aligned} \quad (6)$$

In the absence of disorder, one has $I = \Sigma = 0$ in (6). σ_{xx} vanishes in the dc limit $\omega = 0$ and has a δ-function peak at cyclotron frequency $\omega = \omega_c$. On the other hand, $\sigma_{xy}(\omega = 0) = \rho_c/B$, and σ_{xy} has a pole at $\omega = \omega_c$. In the presence of disorder the crystal is pinned and conductivities develop a new peak at the *pinning frequency* $\omega = \omega_p$. There is also an upward shift of the cyclotron resonance peak from ω_c by a quantity of order ω_p.

To obtain the full frequency dependence of the conductivities, one needs to compute $I(\omega)$. The full solution can be found in [19]. Here we treat the experimentally relevant limit $\omega_c \gg d/\sqrt{\rho_m c}$. A typical plot of $\text{Re}\sigma_{xx}$ is

shown in Fig. 1. Since $I(\omega = 0) = 0$ by definition in the pinned crystal, the dc value of σ_{xx} is still zero but that of σ_{xy} is zero in contrast to the pure case where it was finite. The peaks at the new resonance frequencies have a finite height and width due to disorder-induced dissipation and are determined by Σ and $I(\omega_n)$. Earlier results [12] can be recovered by setting $I(\omega_n) = 0$ in all of the equations. However, the presence of the $I(\omega_n)$ term has many important physical consequences. In the absence of $I(\omega_n)$ the peaks would be δ functions at ω_p^0 and $\omega_c + \omega_p^0$ with $\omega_p^0 = \Sigma/\omega_c$. In contrast, here the peaks are centered around a frequency $\omega_p < \omega_p^0$ and this shift is given by Σ. The peaks have a nontrivial structure and are asymmetric about the resonance frequencies, as can be inferred from (6) and seen in Fig. 1. This invalidates the Lorentzian shape which was used to arbitrarily broaden the δ functions in Ref. [12]. The peaks we obtain are much narrower than the Lorentzian broadened ones.

For frequencies $\omega \ll \omega_p$ and $\omega_p \ll \omega \ll \omega_c$, analytical solutions can be obtained. We find

$$I(\omega_n) = \sqrt{2\rho_m\Sigma + \frac{\pi\rho_m^2\omega_c^2\Sigma^{1/2}}{2\sqrt{cd^2}}}|\omega_n|, \qquad \omega \ll \omega_p\,,$$

$$I(\omega_n) = \frac{\Sigma}{6}\ln\frac{\rho_m^2\omega_n^2\omega_c^2}{d\Sigma^{3/2}}, \qquad \omega_p \ll \omega \ll \omega_c\,. \tag{7}$$

Using (7) in (6), we obtain for $\omega \ll \omega_p$:

$$[\mathrm{Re},\mathrm{Im}]\sigma_{xx}(\omega) = \left[\rho_c^2\sqrt{2\rho_m\Sigma + \frac{\pi\omega_c^2\Sigma^{1/2}}{2\sqrt{cd^2}}}\left(\frac{\omega}{\Sigma}\right)^2, \rho_c^2\frac{\omega}{\Sigma}\right], \tag{8}$$

$$[\mathrm{Re},\mathrm{Im}]\sigma_{xy}(\omega) = \left[\rho_c^2\rho_m\omega_c\left(\frac{\omega}{\Sigma}\right)^2, \rho_c^2\rho_m^{3/2}\frac{\omega_c\omega^3}{\Sigma^{5/2}}\right].$$

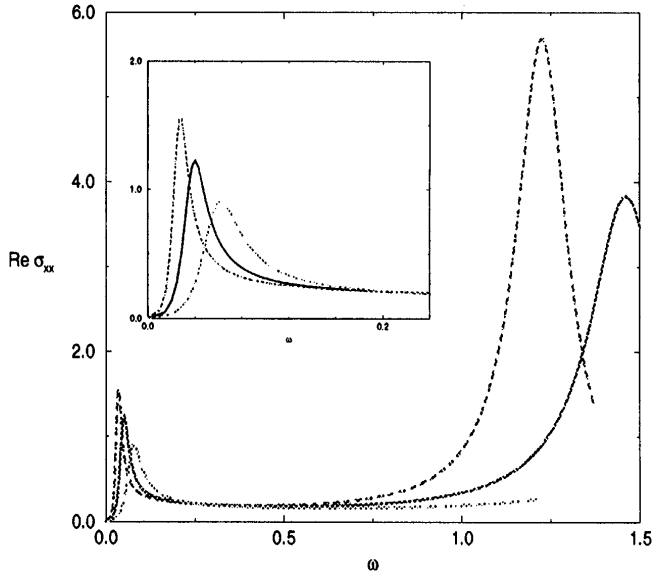

FIG. 1. $\mathrm{Re}\sigma_{xx}(\omega)$ (in units of $b\rho_m/\rho_c$) as a function of ω (in units of $\rho_c^3/\rho_m^2 b$), where $b = c(2\pi^2)^{-1/6}a^2\Delta\rho_m^{-2}(e/\hbar)^3$ for the case $r_f < l_c$ for different values of the field B in units of $\rho_c^2/\rho_m b$. The short-dashed line represents $B = 0.33$, the solid line represents $B = 0.4$, and the long-dashed line represents $B = 0.5$. The inset is a magnification of the pinning peak.

In the region $\omega_p \ll \omega \ll \omega_c$ we find, using (7),

$$\mathrm{Re}\sigma_{xx}(\omega) \sim \frac{\rho_c^2}{\rho_m^2}\frac{\Sigma}{\omega_c^2\omega}, \qquad \mathrm{Re}\sigma_{xy}(\omega) \sim \frac{\rho_c}{B}\,. \tag{9}$$

$\mathrm{Re}\sigma_{xx}$ and $\mathrm{Re}\sigma_{xy}$ are both quadratic in ω. Since the pinned WC has the characteristics proposed for the HI, it seems unnecessary to invoke the existence of the HI as a new phase. Equation (8) can be used to calculate the dielectric constant $\epsilon(\omega) = \mathrm{Im}\sigma_{xx}(\omega)/\omega$. Its dc value is given by $\epsilon = \rho_c^2/\Sigma$ and is also a measure of the characteristic frequency defined by disorder.

Calculating the resistivities $\rho_{\alpha\beta} = -\sigma_{\alpha\beta}/(\sigma_{xx}^2 + \sigma_{xy}^2)$, we find that the pinned crystal is indeed insulating, i.e., $\rho_{xx}(\omega = 0) = \infty$. More importantly, the Hall resisitivity ρ_{xy} turns out to be independent of ω and T and has the same value as that in the pure system $\rho_{xy}(\omega) = B/\rho_c$. A similar result was argued to hold at $T = 0$ [13]. At $T > 0$ it would be necessary to go beyond the GVM approximation to ascertain whether ρ_{xy} still sticks to its classical value. Indeed, the GVM misses solitonlike excitations, which are known to be important for finite T physics [18,20].

It is interesting to calculate the field dependences of the above quantities which are of direct experimental relevance. For reasons of space, we present here the results for the pinning peak only. A similar study can be made for the cyclotron peak [19]. The field dependence of the pinning peak [whose width is naively of $O(\omega_p)$] is governed by Σ whose value is in turn dictated by the relative sizes of r_f and l_c. Traditionally, in the context of CDW, Σ has been related to the Fukuyama-Lee length R_a at which relative displacements are of order a, as $\Sigma \sim R_a^{-2}$. However, for the present case, such a connection does not hold, because disorder can *a priori* vary at scales much smaller than a, unlike in CDW. This is similar to the case of vortex systems where pinning is controlled by the Larkin length R_c, defined as the scale below which the physics of (2) can be described perturbatively by a model where uncorrelated Gaussian random forces of strength $\Delta_f = \sum_{\mathbf{K}} \mathbf{K}^2\Delta_K$ act independently on each electron. Within this model, R_c is given by $\langle[\mathbf{u}(\mathbf{R}_c) - \mathbf{u}(\mathbf{0})]^2\rangle_f = \max[r_f^2, l_c^2] \equiv \xi_0^2$. When $R_c \gg a$, the crystal is pinned collectively. In this regime, using the GVM, one finds $\Sigma = c(2\pi^2)^{-1/6}R_a^{-2}(a/\xi_0)^6$. Here $R_a \sim \rho_m a^2/\sqrt{\Delta}$ and $\xi_0 = \max[r_f, l_c]$. This corresponds to $\Sigma \sim cR_c^{-2}$, implying that $l \sim R_c$ (defined above). The length scale determining the peak in the conductivity is thus R_c and not R_a. It is important to distinguish between these two lengths since R_c can have an explicit dependence on the magnetic field. This yields two very different regimes. One is $r_f < l_c$, which gives $\Sigma = bB^3$ [$b = c(2\pi^2)^{-1/6}a^2\Delta\rho_m^{-2}(e/\hbar)^3$], hence $\omega_p(B) \propto B^2$, and the pinning peak moves up and broadens with increasing field. This is the case illustrated in Fig. 1. The second regime is $r_f > l_c$, leading to a Σ independent of B and $\omega_p(B) \propto B^{-1}$. Thus the pinning peak

TABLE I. Magnetic field dependences of various dynamical quantities.

Regime	$\sum$	ω_p	E_T	ϵ
$r_f > l_c$	B^0	B^{-1}	B^0	B^0
$r_f < l_c$	B^3	B^2	$B^{5/2}$	B^{-3}
$R_c < a$	$B^{3/2}$	$B^{1/2}$	B	$B^{-3/2}$

moves towards the origin and gets narrower with increasing field. For CDW in a magnetic field, $r_f \sim a$ and one is always in the second regime. In these two regimes the height of the pinning peak *decreases* as B^{-1} with increasing field. For $r_f \ll l_c$, ω_p does not increase indefinitely with B, rather there is a crossover to another regime when $R_c \sim a$, where single particle pinning effects are dominant. Here the correspondence between R_c and l, and hence Σ, no longer holds. One then finds that $\Sigma \propto B^{3/2}$ and $\omega_p \propto B^{1/2}$ provided $l_c \gg r_f$. In contrast, the peak at $\omega_c + \omega_p$ always moves upwards with increasing B.

Another important measurable quantity is the threshold electric field E_T, necessary for the crystal to slide. This again shows the interplay between l_c and r_f. Using collective pinning arguments [21], one gets $E_T = cR_c^{-2}\xi_0$ when $R_c \gg a$. The threshold field has the same regimes as above and the field dependences are shown in Table I. Note that, for $r_f > l_c$, E_T is independent of the field (as for CDW). Since both E_T and ω_p are related to R_c, one has $E_T \propto \omega_p$ but with a prefactor depending on ξ_0 and not on a, as given by previous CDW estimates. When $R_c \leq a$, we enter the regime of single particle pinning. Using $\Sigma \propto B^{3/2}$, the threshold field is now $E_T \propto \Sigma l_c \propto B$ [22]. Finally, due to the variation of Σ with the field, the dielectric constant ϵ exhibits the behaviors shown in Table I. Therefore, in addition to detailed frequency measurements of the conductivity, measurements of the B dependences of ϵ could serve as a signature for the WC.

Some of the existing experimental results can be interpreted within our theory. Contrary to previous estimates, it allows a scenario where the pinning frequency *increases* with the field, as was seen in recent experiments [16]. Using the typical values for GaAs hole samples [16] we obtain for the experimentally observed $\omega_p \simeq 1.1$ GHz a value of $R_a \sim 14a$, well in the regime where our theory of collective pinning applies. A simultaneous increase in E_T vs B is observed which is in qualitative agreement with the above predictions. Our theory also predicts that the Hall resistance takes its classical value, which is observed experimentally [7]. However, many problems remain. Experimentally, the peak height in $\sigma(\omega)$ seems to increase with B which we cannot account for at present. Some experiments [7] seem to report a different behavior for the conductivity [24]. Finite temperature and strong disorder effects also need to be understood. Both problems require a careful treatment of the topological defects and solitons, which is beyond the scope of the present study. While a phase transition similar to the one occurring in 3D vortex lattices [11] is unlikely in $d = 2$, one expects a marked crossover between a weakly pinned WC and a strongly pinned one.

In conclusion, we have developed a comprehensive theory for the WC pinned by weak disorder. In addition to detailed frequency dependences of the real and imaginary parts of the conductivities, we have obtained the magnetic field dependences of various dynamical quantities. We find that the magnetic field not only confines the electrons but also plays a crucial role in determining the response of the system to disorder. This dynamical effect, not captured by previous static approximations, allows the possibility of observing novel field dependences.

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[1] E. Wigner, Phys. Rev. **46**, 1002 (1934).
[2] E. Y. Andrei *et al.*, Phys. Rev. Lett. **60**, 2765 (1988).
[3] R. L. Willet *et al.*, Phys. Rev. Lett. **65**, 633 (1990).
[4] F. I. B. Williams *et al.*, Phys. Rev. Lett. **66**, 3285 (1991); F. Perruchot, Ph.D. thesis, Ecole Polytechnique, Paris, 1995.
[5] V. J. Goldman *et al.*, Phys. Rev. Lett. **65**, 2189 (1990).
[6] H. C. Manoharan *et al.*, Phys. Rev. Lett. **77**, 1813 (1996).
[7] T. Sajoto *et al.*, Phys. Rev. Lett. **70**, 2321 (1993).
[8] I. V. Kukushkin and V. B. Timofeev, Phys.-Usp. **36**, 549 (1993).
[9] A. A. Shashkin *et al.*, Phys. Rev. Lett. **73**, 3141 (1994).
[10] S. C. Zhang, S. Kivelson, and D. H. Lee, Phys. Rev. Lett. **69**, 1252 (1992).
[11] T. Giamarchi and P. Le Doussal, Phys. Rev. B **52**, 1242 (1995).
[12] H. Fukuyama and P. Lee, Phys. Rev. B **18**, 6245 (1978).
[13] B. G. A. Normand, P. B. Littlewood, and A. J. Millis, Phys. Rev. B **46**, 3920 (1992); X. Zhue, P. B. Littlewood, and A. J. Millis, *ibid.* **50**, 4600 (1994).
[14] U. Wulf, J. Kucera, and E. Sigmund, Phys. Rev. Lett. **77**, 2993 (1996).
[15] Y. P. Li *et al.*, Solid State Commun. **95**, 619 (1995).
[16] C. C. Li *et al.*, Phys. Rev. Lett. **79**, 1353 (1997); F. I. B. Williams (private communication).
[17] L. Bonsall and A. A. Maradudin, Phys. Rev. B **15**, 1959 (1977).
[18] T. Giamarchi and P. Le Doussal, Phys. Rev. B **53**, 15 206 (1996).
[19] R. Chitra, T. Giamarchi, and P. Le Doussal (to be published).
[20] A. I. Larkin and P. A. Lee, Phys. Rev. B **17**, 1596 (1978).
[21] A. I. Larkin and Y. M. Ovchinnikov, J. Low Temp. Phys. **34**, 409 (1979).
[22] This holds because here the "single particle localization length" (see Refs. [11,23]) is $l_\perp \sim l_c$. There could be an additional regime where $r_f \ll l_\perp \ll l_c$, where arguments similar to [23] can be made.
[23] D. R. Nelson and V. M. Vinokur, Phys. Rev. B **48**, 13 060 (1993).
[24] They have reported $\mathrm{Re}\sigma_{xx}$ and $\mathrm{Im}\sigma_{xx} \propto \omega$. However, this linear behavior in both $\mathrm{Re}\sigma_{xx}$ and $\mathrm{Im}\sigma_{xx}$ violates analyticity properties, and more work is needed to clarify this point before a comparison with theory can be made.