Exact Coherent States of a One-Dimensional Quantum Fluid in a Time-Dependent Trapping Potential

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We find exact coherent states for a one-dimensional quantum fluid interacting by an inverse-square pair potential, contained by a time-dependent harmonic trapping potential. These states are those that would evolve from the ground state of the time-independent problem. Correlations are determined, and a hydrodynamic description is shown to be exact. We treat the case of a nondissipative "sloshing" mode characteristic of superfluidity, free expansion where time-of-flight measurements do not give the momentum distribution, and a periodically varying trapping potential exhibiting alternating regions of stable and unstable behavior. [S0031-9007(98)05857-8]

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Bose-Einstein condensation in a noninteracting gas was predicted many years ago [1], and the superfluid transition, first seen in the strongly interacting ^{4}He system [2], confirms our basic understanding of this phase transition. However, as has often been emphasized, the superfluid transition and the finite occupation of the zero momentum state—the Bose-Einstein condensation—are distinct phenomena. Therefore, it would be of interest to observe experimentally the original Bose-Einstein condensation as a delta-function peak in the momentum distribution. Such a peak should be most evident in the ground state at zero temperature, and should even exist for systems of reduced dimensionality [3].

Recently, evidence for such an "ideal" Bose-Einstein condensation has been observed [4] in a cloud of approximately 200 ^{87}Rb atoms cooled to temperatures as low as 20 nK. Signatures of the Bose-Einstein condensation were seen by relaxing the magnetic trap, allowing the atoms to expand freely, and then imaging the cloud—and, hence, the velocity distribution of the atoms in the cloud—by laser light. One of the purposes of this paper is to point out by a simple soluble example that the link between the velocity distribution seen and the momentum distribution inferred is indirect at best.

In this Letter, we consider the exactly soluble [5] example of a one-dimensional system of identical particles interacting by an inverse-square pair potential. This system must be confined or trapped by some external potential; we choose a harmonic well with a time-dependent spring parameter. In the following, we will explore this simple system in detail. We first show that the most important time-dependent coherent states—those that evolve from the ground state for a time-independent trap—can be found exactly. There is no restriction on the time rate of change of the harmonic trapping potential. Then we review the previous exact results for the ground state of such a system in a time-independent trap. For the time-dependent trap, the correlations are determined, and a hydrodynamic description is shown to hold exactly in all situations. We then briefly treat three important cases: a nondissipative "sloshing" mode, which indicates a type of superfluidity; free expansion, where time-of-flight measurements are shown to not give the momentum distribution; and a system driven by a periodically varying trapping potential, which exhibits alternating regions of stable and unstable behavior. A final section summarizes and discusses the significance of our results.

We consider a one-dimensional N-body system obeying the time-dependent Schrödinger equation

$$\frac{1}{2\Psi}\sum_{j=1}^{N}\frac{\partial^2\Psi}{\partial x_j^2} + i\frac{1}{\Psi}\frac{\partial\Psi}{\partial t} = V(t) = \sum_{j>i=1}^{N}\frac{\lambda(\lambda-1)}{(x_j-x_i)^2} + \frac{K(t)}{2}\sum_{j=1}^{N}x_j^2 \equiv V_2 + V_1(t)\,. \tag{1}$$

We now and, henceforth, use units in which $m = \hbar = 1$. The potential $V(t)$ consists of V_2, a time-independent inverse-square pair potential of strength $\lambda(\lambda-1)$, and $V_1(t)$, a time-dependent one-body harmonic trapping potential with a spring parameter $K(t)$. We make the following ansatz for the solution [5]:

$$\Psi = \psi(t)^{-N[1+\lambda(N-1)]/2}\prod_{j=1}^{N}e^{-\omega(t)x_j^2/2}\prod_{j>i=1}^{N}|x_j-x_i|^{\lambda}. \tag{2}$$

Since the wave function vanishes whenever two particles touch, the statistics of the particles can be put in "by hand." Upon evaluation, we find that the complex function of time $\omega(t)$ must obey the differential equation

$$\omega^2(t) - i\frac{d\omega(t)}{dt} = K(t)\,. \tag{3}$$

The remaining differential equation for $\psi(t)$ is

$$\frac{1}{\psi(t)}\frac{d\psi(t)}{dt} = i\omega(t)\,. \tag{4}$$

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This second equation provides both a time-dependent phase and ensures that the wave function remains normalized.

Equation (3) for $\omega(t)$ is the well-known Riccati equation, and (4) is a familiar change of variables to a function $\psi(t)$ which satisfies the equivalent linear equation

$$\frac{d^2\psi(t)}{dt^2} + K(t)\psi(t) = 0. \tag{5}$$

Writing $K(t) = K_0 - K_1(t)$, this equation is much like the time-independent Schrödinger equation with a potential K_1 and energy eigenvalue K_0. Suppose the equations are satisfied with $\omega(t)$, $\psi(t)$, and $K(t)$. Then $\alpha\omega(t\alpha)$, $\psi(t\alpha)$, and $\alpha^2 K(t\alpha)$ also satisfy the equations. Thus, if $K(t)$ has a characteristic time τ or frequency ν, then we can put the problem in a standard form by choosing the scale factor α to be τ or $1/\nu$.

There are some physical requirements on $K(t)$. First, it should be real. Second, $K(t)$ should be non-negative so that the system is stable. This is particularly significant when we examine Eq. (5) for the function $\psi(t)$. Writing $K(t) = K_0 - K_1(t)$, the condition $K(t) \geq 0$ becomes $K_0 \geq \min[K_1(t)]$, a reasonable requirement for the Schrödinger equation. Let $\psi_{1,2}(t)$ be any two independent solutions to (5). Then the general solution for $\omega(t)$ will be given by

$$\omega(t) = -i\,\frac{A\dot{\psi}_1 + B\dot{\psi}_2}{A\psi_1 + B\psi_2} = -i\,\frac{\dot{\psi}_1 + e^{-\phi-i\theta}\dot{\psi}_2}{\psi_1 + e^{-\phi-i\theta}\psi_2}. \tag{6}$$

Here we have parametrized the complex number B/A as $B/A = e^{-\phi-i\theta}$; this gives us two independent real parameters to fit the boundary conditions for $\omega(t)$. Particular solutions will be investigated later. Here, we simply remark that the time-independent case is given by $\psi(t) = e^{it\sqrt{K}}$.

We begin by reviewing the results for a time-independent harmonic well with spring constant $K = \omega^2$, ω real. The ground state energy is given by $E = N\omega[1 + \lambda(N-1)]/2$. We can decompose this into kinetic, pair-potential, and trapping-potential energy by the Feynman-Hellmann theorem; exactly half the total energy is from the harmonic well, a sort of "equipartition theorem."

The diagonal correlations are independent of statistics. We first evaluate the number density $d(x\,|\,\omega)$ as

$$d(x\,|\,\omega) = \begin{cases} \dfrac{2N}{\pi x_0}\sqrt{1 - x^2/x_0^2}, & x^2 \leq x_0^2; \\ 0, & x^2 \geq x_0^2 \equiv 2N\lambda/\omega. \end{cases} \tag{7}$$

This density serves as the local thermodynamic variable and allows a hydrodynamic description, so that the total chemical potential is constant, or $\mu = \mu(d) + \omega^2 x^2/2$. Comparing this with the expression (7) then allows us to determine the thermodynamics of the bulk inverse-square fluid through $\mu(d) = \pi^2\lambda^2 d^2/2$, implying an energy density $\varepsilon(d) = \pi^2\lambda^2 d^3/6$. All local properties such as total energy density, pair correlation function, and momentum distribution are then those of the inverse-square fluid at the local density $d(x\,|\,\omega)$. These all scale simply with density.

One finds (i) for bosons, the momentum distribution $n(p)$ has a divergence at zero momentum, given by $n(p) \to |p|^{\alpha_b(\lambda)}$, as $|p| \to 0$, with $\alpha_b(\lambda) = \lambda/2 - 1$; (ii) for (spinless) fermions there is a singularity at the Fermi momentum $p_f = \pi d$ of the form $n(p) \to |p - p_f|^{\alpha_f(\lambda)}\mathrm{sgn}(p - p_f)$ as $|p - p_f| \to 0$, where $\alpha_f(\lambda) = (\lambda + 1/\lambda)/2 - 1$.

We divide $\omega(t) = \omega_r(t) + i\omega_i(t)$ into real and imaginary parts. For the diagonal correlations, only the real part of ω enters, since

$$\Psi^*(x)\Psi(x) = |\psi(t)|^{-N[1+\lambda(N-1)]} \times \prod_{j=1}^{N} e^{-\omega_r(t)x_j^2} \prod_{j>i=1}^{N} |x_j - x_i|^{2\lambda}, \tag{8}$$

and this expression is independent of statistics. Further, all diagonal correlations are exactly the same as for the time-independent case, with the time-dependent $\omega_r(t)$ replacing the time-independent ω. Thus, the local density of the system is given by

$$d(x,t) = d(x\,|\,\omega_r(t)) = \begin{cases} \dfrac{2N}{\pi}\sqrt{x_0^2(t) - x^2}/x_0^2(t), & x^2 \leq x_0^2(t); \\ 0, & x^2 \geq x_0^2(t) \equiv 2N\lambda/\omega_r(t). \end{cases} \tag{9}$$

The boundary $x_0(t)$ moves in time as $x_0(t) = \sqrt{2N\lambda/\omega_r(t)}$. Other correlations are exactly as in the stationary problem with $\omega_r(t)$ replacing ω.

For the off-diagonal correlations, we must include the imaginary part of $\omega(t)$, and so

$$\Psi^*(y)\Psi(x) = |\psi(t)|^{-N[1+\lambda(N-1)]} \times \prod_{j=1}^{N} e^{i\omega_i(t)(y_j^2 - x_j^2)/2 - \omega_r(t)(y_j^2 + x_j^2)/2} \times \prod_{j>i=1}^{N} |(x_j - x_i)(y_j - y_i)|^{\lambda}. \tag{10}$$

Thus, integrating over all variables but one, if $\rho_1(rd(x)\,|\,\lambda)$ denotes the $(x + r/2, x - r/2)$ element of the one-particle density matrix of the wave function for the time-independent inverse-square potential, then the actual local, time-dependent one-particle density matrix is $\rho_1(r;x,t) = e^{i\omega_i(t)xr}\rho_1(rd(x,t))$. Thus, when we calculate the momentum distribution $n(k;x,t)$ at point x and time t, by Fourier transforming with respect to the difference r, we find it to be given by the momentum distribution of the stationary problem, scaled by the local density $d(x,t)$ and centered at a shifted origin given by the local average velocity $v(x,t) = -x\omega_i(t)$.

This local average velocity increases linearly as a function of x to a maximum at the boundary given by $v_0(t) = v(x_0, t) = -\omega_i(t)\sqrt{2N\lambda/\omega_r(t)}$.

The results that we have derived by exact calculation suggest the following hydrodynamic description. At each point in space and time, there is a local average density $d(x,t)$ and local average velocity $v(x,t)$. These have a position and time dependence given by

$$d(x,t) = \frac{2N}{\pi x_0(t)}\sqrt{1 - [x/x_0(t)]^2}, \qquad v(x,t) = \dot{x}_0(t)\frac{x}{x_0(t)}. \tag{11}$$

Using the equation of motion for $\omega(t)$, one can verify that the continuity equation $\partial d(x,t)/\partial t + \partial j(x,t)/\partial x = 0$ is satisfied, with the current $j(x,t)$ given by $j(x,t) = d(x,t)v(x,t)$. Also, using the expression for the chemical potential, one can verify the hydrodynamic equation of motion in the form

$$\frac{Dv(x,t)}{Dt} = \left[\frac{\partial}{\partial t} + v(x,t)\frac{\partial}{\partial x}\right]v(x,t) = -\frac{\partial}{\partial x}\left[\mu(x,t) + K(t)x^2/2\right]. \tag{12}$$

The last expression follows from the expression for the thermodynamics of the inverse-square fluid. These equations are also sufficient to determine $d(x,t)$ and $v(x,t)$, subject to the given initial or boundary conditions. Locally, the fluid is otherwise in equilibrium, and so all other thermodynamic quantities are the same as in equilibrium, apart from appropriate scaling by $d(x,t)$ and $v(x,t)$. Thus, the fluid behaves like an ideal Euler fluid, without dissipation.

Before we treat a time-dependent spring parameter K, let us first return and reconsider the fluid confined in a time-independent harmonic well. In addition to the time-independent ground state wave function, there is a time-dependent coherent sloshing mode, which we can determine by returning to Eq. (5) for $\psi(t)$. Since this is the Schrödinger equation for a free particle, writing $K = k^2$, we immediately find the solution from Eq. (6) to be

$$\omega(t) = k\frac{e^{ikt} - e^{-\phi - i(kt+\theta)}}{e^{ikt} + e^{-\phi - i(kt+\theta)}} = k\frac{\sinh\phi + i\sin(2kt+\theta)}{\cos(2kt+\theta) + \cosh\phi}. \tag{13}$$

The frequency is $2k = 2\sqrt{K}$ independent of amplitude, and θ is the phase of the oscillation.

The physical situation we now have in mind is when the system has been trapped in a time-independent harmonic well with spring constant K_0 for a long time in the past. Then around time $t = 0$, the trapping potential is "turned off," over a time interval of order τ. Thus, $K(t)$ is to be something like $K(t) = K_0[1 - \tanh(t/2\tau)]/2 = K_0/(1 + e^{t/\tau})$. The parameter τ can be scaled away, so with $k = \tau\sqrt{K_0}$, and t/τ replaced by t, the equation for ψ is in the dimensionless form $-\ddot{\psi} + k^2\psi/(1 + e^{-t}) = k^2\psi$. This is exactly of the form of a standard quantum mechanics problem; see, for instance, the text of Landau and Lifshitz. Our boundary condition is that the system is initially in the ground state, or $\psi(t) \to e^{ikt}$, as $t \to -\infty$. In the corresponding quantum problem, this means that the energy is exactly equal to the height of the step potential—a special case. The general solution of (5), which has the correct asymptotic form can be written in terms of the hypergeometric function $F[\alpha, \beta; \gamma; x]$ as $\psi(t) = e^{ikt}F[ik, ik; 1 + 2ik; -e^t]$.

The parameter k serves as the dimensionless scaling parameter. The two limits are (i) when the expansion is *slow*, then $k \to \infty$, and so $\omega(t) \approx \omega_r(t) \approx \sqrt{K(t)}$; (ii) in the other limit of a *fast* expansion, when $k \to 0$, then $\omega(t) \approx \sqrt{K_0}/(1 + it\sqrt{K_0})$.

When the system undergoes a rapid free expansion, not only is the energy conserved, but so are all the other integrals of motion. Thus, the asymptotic momenta are determined by the asymptotic Bethe ansatz. A region of the fluid $x \to x + dx$ with density $d(x)$ contributes asymptotic momenta or velocities with a density $dx\rho(k;x)$, where $\rho(k;x)$ is given as [5]

$$\rho(k) = \begin{cases} 1/2\pi\lambda, & |k| < \pi\lambda d; \\ 0, & |k| > \pi\lambda d. \end{cases} \tag{14}$$

Then the total asymptotic momentum density is obtained by integrating over the fluid, giving $2N\sqrt{1 - (k/k_0)^2}/\pi k_0$, with $k_0 = 2\lambda N/x_0 = \sqrt{2\lambda N\omega_0}$, and $\omega_0 = \sqrt{K_0}$. By time of flight, this leads to a final expanding density of the form $d(x,t) = 2N\sqrt{1 - x^2/x_0^2(t)}/\pi x_0(t)$, with $x_0(t) = k_0 t$. Using the result for a rapid expansion in (9), these two results are exactly the same.

With a time-dependent trapping potential driving the fluid, we have the possibility of a very rich dynamical behavior. As an example, we can drive the system with a spring parameter periodic in time with period $2\pi/\nu$, an example being $K(t) = K_0 - 2K_1\cos(2\nu t)$. Once again, the parameter ν can be scaled away, so with $a = K_0/\nu^2$, $q = K_1/\nu^2$, and νt replaced by t, Eq. (5) for ψ is in the dimensionless form

$$-\frac{d^2\psi(t)}{dt^2} + 2q\cos(2t)\psi(t) = a\psi(t). \tag{15}$$

This is the standard form for Mathieu's equation. However, everything we have to say in what follows holds for a general periodic spring parameter.

Let us define $\psi_1(t)$ to be the real even solution normalized so that $\psi_1(0) = 1$, and $\psi_2(t)$ to be the real odd solution normalized so that $\dot{\psi}_2(0) = 1$. Then we write a general solution with $\psi(0) = A_0$ and $\dot{\psi}(0) = B_0$, as $\psi(t) = A_0\psi_1(t) + B_0\psi_2(t)$ for $0 \le t \le \pi$. When we get to $\pi \le t \le 2\pi$, we start all over again, writing $\psi(t) = A_1\psi_1(t - \pi) + B_1\psi_2(t - \pi)$ for $\pi \le t \le 2\pi$. The new coefficients are related to the old through

$A_1 = A_0\psi_1(\pi) + B_0\psi_2(\pi)$ and $B_1 = A_0\dot{\psi}_1(\pi) + B_0\dot{\psi}_2(\pi)$. In this way, we go on forever, with

$$\begin{pmatrix} A_n \\ B_n \end{pmatrix} = \begin{pmatrix} \psi_1(\pi) & \psi_2(\pi) \\ \dot{\psi}_1(\pi) & \dot{\psi}_2(\pi) \end{pmatrix} \begin{pmatrix} A_{n-1} \\ B_{n-1} \end{pmatrix}. \quad (16)$$

Evaluating $\omega(t)$ only at the points $t = n\pi$ from Eq. (6), we find

$$\omega(n\pi) \equiv \omega_n = i\,\frac{B_n}{A_n} = i\,\frac{\dot{\psi}_1(\pi) - i\dot{\psi}_2(\pi)\omega_{n-1}}{\psi_1(\pi) - i\psi_2(\pi)\omega_{n-1}} \equiv i\,\frac{b - ia\omega_{n-1}}{d - ic\omega_{n-1}}.$$

Since $\psi_1(t)\dot{\psi}_2(t) - \psi_2(t)\dot{\psi}_1(t)$ is constant, and so equal to 1, then $ad - bc = 1$. Since a, b, c, and d are real, this transformation is equivalent to a Möbius transformation; such transformations form the group of proper isometries of the hyperbolic plane in the Poincaré representation of hyperbolic geometry. They also map the right half of the complex plane onto itself. Under the action of the transformation, ω moves along invariant flow lines. The nature of these flow lines depends on the nature of the transformation: (i) If the transformation is elliptic with $|a + d| < 2$, then flow lines are closed circles enclosing the complex fixed point in the right half plane; (ii) if the transformation is hyperbolic with $|a + d| > 2$, then flow lines are segments of circles in the right half plane connecting the two imaginary fixed points. For the equivalent Schrödinger equation, the energy is either in an allowed or forbidden band, respectively. For the wave function to be normalizable, $\omega(t)$ must have a positive real part, and so case (ii) is unstable.

To summarize, we have carried out the program we outlined in the introduction. We have solved very explicitly a one-dimensional example of a time-dependent interacting quantum many-body system; there are very few other such examples. We found important time-dependent coherent states, correlations in these states were determined, and a hydrodynamic description was shown to hold exactly in all situations. We then examined three important cases: (i) a nondissipative sloshing mode, (ii) the free expansion of the system, and (iii) the system driven by a periodically varying trapping potential. The behavior of this system is interesting. The system exhibits a persistent sloshing mode, which one might argue is evidence of a form of one-dimensional superfluidity. The correlation functions exhibit power law decay, similar to the behavior found for the Kosterlitz-Thouless transition. The system exhibits a suppression of condensate fragmentation and phase coherence, as emphasized by Nozierès [3].

Now, let us list the peculiarities of the system. First, the system is one dimensional, so most of the interesting physics must happen at zero temperature. Second, the system is integrable; this is what allows us to solve it. One might then argue that the persistent sloshing mode is the result of the system being integrable with a large number of conservation laws. Third, the system is long ranged. In fact, the potential and kinetic energies scale in exactly the same way, so that the strength of the interaction is dimensionless. All of these points are interrelated, and the system gives us the opportunity to explore such questions in detail, and perhaps arrive at some answers.

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