Random Walks, Reaction-Diffusion, and Nonequilibrium Dynamics of Spin Chains in One-Dimensional Random Environments

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(Received 24 October 1997)

Sinai's model of diffusion in one dimension with random local bias is studied by a real space renormalization group which yields asymptotically exact long time results. The distribution of the position of a particle and the probability of it not returning to the origin are obtained, as well as the two-time distribution which exhibits "aging" with $\ln t / \ln t'$ scaling and a singularity at $x(t) = x(t')$. The effects of a small uniform force are also studied. Extension to motion of many domain walls yields nonequilibrium time dependent correlations for the 1D random field Ising model with Glauber dynamics and "persistence" exponents of 1D reaction-diffusion models with random forces. [S0031-9007(98)05800-1]

PACS numbers: 64.60.Ak, 75.10.Hk

The development of order in systems with a broken symmetry is of interest in many contexts. "Coarsening" of domain structures evolving towards equilibrium has been studied extensively [1]. But little is known analytically about domain growth in the presence of quenched disorder [2,3]. Nevertheless, phenomenological descriptions of the nonequilibrium dynamics of various random magnetic systems have been developed in terms of "droplets" separated by domain walls [3]. Because of the very slow dynamics associated with activation over large free energy barriers, even the apparent equilibrium properties of these systems are dominated by the nonequilibrium dynamics, as also occurs in infinite-range models [3,4]. Even in one dimension some random systems exhibit ultraslow growth and aging phenomena. Exact results in 1D could thus be used as testing grounds for more complex $D > 1$ cases which have resisted analytic attack.

In this Letter we study the diffusion of a particle in a 1D random potential which itself has the statistics of a 1D random walk [5]. Extensions to many interacting particles allow us to study, via domain walls, the Glauber dynamics of 1D Ising models, in particular random field ferromagnets and spin glasses in a magnetic field. This leads also to the consideration of more general diffusion-reaction processes in such energy landscapes. Various analytic results are known for the single particle model (Sinai model) [5–11] but these primarily concern single time quantities. Here we use a real space renormalization group (RSRG) method related to that used to study disordered quantum spin chains [12–14]. This allows us to compute a host of quantities such as first passage (persistence) exponents, single time correlations, and even two time correlations that are probed in aging experiments. Despite its approximate character, the RSRG yields results that are asymptotically exact at long times.

The model is defined as follows: Particles diffuse on a 1D lattice in a potential U_i, with i the site index. A "force" variable $f_i \equiv U_i - U_{i+1}$ is defined on each bond $(i, i+1)$ with these f_i independent random variables. Since one can group together neighboring bonds with the same sign of the force, we study with no loss of generality, a "zigzag" potential (see Fig. 1) with the f_i alternatively positive and negative but with a distribution of "bond" lengths l_i. Our model is thus defined by $f_i = (-1)^{i+1} F_i$, where the positive $F_i = |U_i - U_{i+1}|$, which are effectively energy barriers, are the natural variables. The pairs of bond variables F, l are chosen independently from bond to bond from a distribution $P(F, l)$. In the presence of a directionality bias one needs two distinct distributions $P(F, l)$ for "descending bonds" and $R(F, l)$ for "ascending bonds."

We are interested in long times when the behavior will be dominated by large barriers and it is on these that we must focus. Our RG procedure is conceptually simple: in a given energy landscape it consists of iterative decimation of the bond with the *smallest barrier*, say $F_2 = U_3 - U_2$, as illustrated in Fig. 1. At time scales much longer than

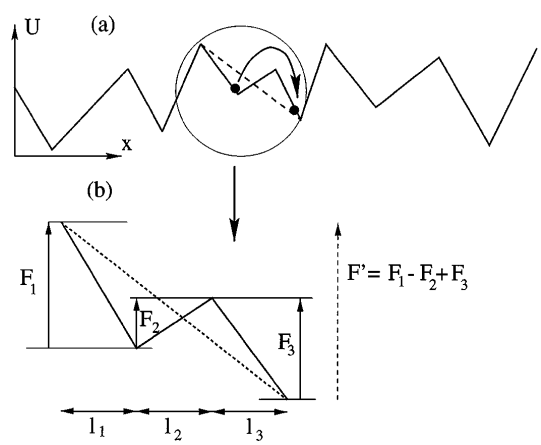

FIG. 1. (a) Energy landscape in Sinai model (b) decimation method: the bond with the smallest barrier $F_{\min} = F_2$ is eliminated resulting in three bonds being grouped into one.

$\exp(F_2/T)$, local equilibrium will be established between sites 2 and 3 and the rate for the walker to get from 4 to 1 will be essentially the same as it would be if sites 2 and 3 did not exist but 1 and 4 were instead connected by a bond with barrier $F' = F_1 - F_2 + F_3$ and length $l' = l_1 + l_2 + l_3$. We thus carry out exactly this replacement which preserves the zigzag structure and the larger scale extrema of the potential. With Γ defined to be the smallest remaining barrier at a given stage of the RG, we eliminate the barriers in the range $\Gamma < F < \Gamma + d\Gamma$. The new variables are *independent* from bond to bond. Introducing $\zeta \equiv F - \Gamma > 0$ one finds the following RG flow equations for the probabilities $P(\zeta, l, \Gamma)$ and $R(\zeta, l, \Gamma)$:

$$\begin{aligned} (\partial_\Gamma - \partial_\zeta)P &= R(0,\cdot) *_l P *_{\zeta l} P + (P_0^\Gamma - R_0^\Gamma)P\,, \\ (\partial_\Gamma - \partial_\zeta)R &= P(0,\cdot) *_l R *_{\zeta l} R + (R_0^\Gamma - P_0^\Gamma)R\,. \end{aligned} \quad (1)$$

Here $*_\zeta$ denotes a convolution with respect to ζ only and $*_{\zeta,l}$ with respect to both ζ and l and we define $P_0^\Gamma \equiv \int_0^\infty dl\, P(\zeta = 0, l, \Gamma)$ and similarly for R_0^Γ. The dynamics implied by this RG is rather simple. Making the obvious identification of $\Gamma = T \ln(t/t_0)$ from Arrhenius dynamics, we see that at very long time the renormalized landscape consists entirely of deep valleys separated by high barriers. A good approximation to the long time dynamics is thus to place the walker *at the bottom* of the renormalized valley at scale $T \ln t$ in which it was initially, since, with high probability, it will be near to that point [5]. Upon proper rescaling of space and time this becomes exact as Γ tends to ∞ as was proven in Ref. [5] for the unbiased case. It remains valid in the weakly biased case in the limit that the bias parameter that controls the long time properties, μ, defined implicitly for the original model with unit length bonds by $\langle \exp(-\mu f_i/T)\rangle = 1$, is very small (see [8,9]).

The RG equations (1) are identical to those derived for the random transverse field Ising chain (RTFIC) in [13] with the identification of $\ln h_k = F_{2k}$ and $\ln J_k = F_{2k+1}$ in the RTFIC with the ascending and descending barriers, respectively [15]. Thus duality in the RTFIC corresponds to reversing the average force. Criticality then corresponds to the zero drift case, while the Griffiths phase in the RTFIC [13] corresponds to the biased phase with zero velocity [8,9]. The deviation from criticality parameter [13] $\delta \equiv (\langle \ln h\rangle - \langle \ln J\rangle)/[\mathrm{var}(\ln h) + \mathrm{var}(\ln J)]$ is analogous at small δ to $\mu/2$.

We consider first the long time dynamics of a single particle, starting with the symmetric, zero bias, case that has the same distribution for all the bonds; i.e., $R = P$. For large Γ, the distribution flows to one of a family of scaling solutions of the RG equations (1). The rescaled probability $\tilde{P}(\eta, \lambda) = (\Gamma^3/\sigma)P(\eta\Gamma, \lambda\Gamma^2/\sigma, \Gamma)$ in terms of the rescaled variables $\eta \equiv \zeta/\Gamma$, $\lambda \equiv l\sigma/\Gamma^2$, when Laplace transformed in λ to s, is found to be [13] $\tilde{P}(\eta, s) = [\sqrt{s}/\sinh(\sqrt{s})]\exp[-\eta\sqrt{s}\coth(\sqrt{s})]$. The average bond length is $\bar{l} = \frac{1}{2\sigma}\Gamma^2$ and we recover the scaling $x \sim \ln^2 t$ [5]. The large scale variance of the potential $\langle (U_i - U_j)^2\rangle \approx 2\sigma|l_{i-j}|$, with l_{i-j} the distance from i to j is conserved by the RG, fixing σ.

The fact that the renormalized barrier distribution becomes infinitely broad is the source of the exactness of our long time results. At late stages of the RG, the chances that two neighboring bonds have F's that are within order T of each other tends to zero for large Γ. Thus substantial errors that are introduced by assigning a particle to one of two almost-equal-depth neighboring valleys rather than splitting its distribution between the two valleys will occur rarely at long scales. Furthermore, any such error is wiped out by a later decimation which eliminates the two valleys in favor of a deeper valley. Since in a deep renormalized valley, the particle tends to be very close to the bottom on the scale of $\bar{l}(\Gamma)$, the rigorous results [5,7] imply that we can obtain the scaled distribution of the position of a particle at time t that started at the origin at time zero, directly from $P(l, \Gamma = T\ln t)$. We henceforth set $T = 1$ and measure distances in units such that $\sigma = 1$. The dynamics "within" a bond and errors in early stages of the RG will only change the microscopic cutoff time t_0.

The renormalized dynamics corresponds to moving the particle from its starting point (distributed uniformly on a bond) to the lower-potential end of the bond. The distribution of its position at time t averaged over the ensemble of random potentials is thus $\overline{\mathrm{Prob}(x, t\,|\,0,0)} = \frac{1}{2}\int_{|x|}^\infty dl\, P(l,\Gamma)/\bar{l}(\Gamma)$. With $\Gamma = \ln t$, it takes the scaling form $\overline{\mathrm{Prob}(x, t\,|\,0,0)} = \frac{1}{\ln^2 t} q(\frac{x}{\ln^2 t})$ where

$$q(X) = \frac{4}{\pi}\sum_{n=0}^\infty \frac{(-1)^n}{2n+1} e^{-\frac{1}{4}\pi^2|X|(2n+1)^2}. \quad (2)$$

With σ reinserted, this coincides with Kesten's rigorous result [7] for a Brownian potential, as it should [5].

But we can now generalize to the biased case with a small average potential drop per unit length $2\delta > 0$. The RG flows Eq. (1) now involve the two distributions R and P. The asymptotic behavior of these flows was found in [13]; it obtains in the scaling limit that Γ is large, while $\lambda = l/\Gamma^2$ and $\gamma \equiv \Gamma\delta$ are both fixed but arbitrary. In terms of the Laplace transform from $|X| \equiv |x|/\Gamma^2$ to s, we obtain for the generalization of Eq. (2),

$$q(X,\gamma) = \left(\frac{\gamma}{\sinh\gamma}\right)^2\left[\theta(X)LT^{-1}\frac{1}{s}\left(1 - \frac{\kappa e^{-\gamma}}{\kappa\cosh\kappa - \gamma\sinh\kappa}\right) + \theta(-X)LT^{-1}\frac{1}{s}\left(1 - \frac{\kappa e^{\gamma}}{\kappa\cosh\kappa + \gamma\sinh\kappa}\right)\right], \quad (3)$$

with $\kappa \equiv \sqrt{s + \gamma^2}$ and the two terms arising from descending and ascending renormalized bonds, respectively. In the limit of small γ, the behavior reduces to the symmetric case Eq. (2). But for large γ, i.e., $\ln t \gg 1/\delta$, the distribution is heavily concentrated to the right of the origin $\overline{\mathrm{Prob}(x, t\,|\,0,0)} \approx \theta(x)\exp(-x/\overline{x(t)})/\overline{x(t)}$ with the mean displacement $\overline{x(t)} \approx t^{2\delta}/(4\delta^2)$, consistent with the small μ limit of the known "Levy front" $L_\mu(t/(\mu^2 x)^{1/\mu})$ [6,9], although the exponent μ of the anomalous diffusion $x \sim t^\mu$ is correct only to leading order in δ, due to corrections to scaling neglected in our RG. We find that the model renormalizes onto a directed model with traps (ascending bonds) of "release time" distribution $\rho(\tau) \sim \tau^{-(1+\mu)}$ [9].

Our method also enables us to compute two time quantities, e.g., $B(x,t,x',t') \equiv \overline{\text{Prob}(xt|x't'|00)}$ which contains information about the dynamics after the system has "aged" from $t = 0$ to t', but the full calculation and result [16] are too complicated to reproduce here. In the regime $\Gamma = \ln t$ and $\Gamma' = \ln t'$ large with $\alpha \equiv \ln t/\ln t'$ a fixed number, we obtain a scaling form $B \approx B(\alpha, \tilde{x}, \tilde{x}')$ in the rescaled variables $\tilde{x} = x/\Gamma^2$ and $\tilde{x}' = x'/\Gamma^2$. Our two time correlations thus exhibit a $\ln t'/\ln t$ aging regime, as found numerically in [11] and argued in [3] for spin glasses in higher dimensions. Interestingly, the rescaled distribution has a delta function component at $\tilde{x} = \tilde{x}'$, of weight $f(\alpha)$, suggested in [11], that we have obtained analytically [16]. It arises from bonds not decimated between t' and t, i.e., from particles staying within the same valley. The mean square additional displacement between t' and t can be obtained from our general results. For large $\alpha = \ln t/\ln t'$, $\langle |x(t) - x(t')|^2 \rangle \approx \frac{61}{180}(\ln t)^4$ independent of the motion up to time t'. But when $\ln t$ and $\ln t'$ are not too separated, i.e., $\alpha \approx 1$, the mean square additional displacement is only $\approx (\ln t')^4 \frac{272}{315}(\ln t/\ln t' - 1)$. In this regime of times the particle is typically trapped in a deep well, but there is a probability of order $(\Gamma - \Gamma')/\Gamma'$ that one of the barriers of the well at time t' is less than Γ. If so, the walker will jump to the bottom of a deeper valley a distance of order $\bar{l}(\Gamma') \sim \Gamma'^2 \approx \Gamma^2$ away yielding the above result. Note, however, that there is a subtle limit implicit here: fixed $\alpha > 1$ as $\ln t \to \infty$ implies that no matter how close α is to one, $t \gg t'$. The physics when t and t' are much closer together is quite different, in particular, when $(t - t')/t' = O(1)$ or less. With probability that approaches one as $t' \to \infty$, in this regime there will be *no* jumps to a new (deeper) valley between t' and t and the additional displacement will typically be small. But its mean square will be dominated by rare configurations—absent in the scaling limit—in which the valley at time t has two almost degenerate minima. Jumping between such minima within a valley persists even for $t \to \infty$ with $t - t'$ fixed and in this limit the statistics of the resulting infinitely deep valley potential becomes that of a random walk restricted to have $U_i - U_{\text{valley-min}} > 0$ [5,11,16].

We now turn to problems involving many walkers. The Glauber dynamics of the 1D (classical) random field Ising model corresponds to *two types* of domain walls A and B which see *opposite* random potentials with the forces f_i being simply twice the corresponding random fields on the dual lattice sites. When the random fields are much smaller than the exchange J, the long time behavior for $T \ll J$ will be universal. We focus on the evolution starting from random initial conditions—e.g., after a quench from a high temperature. At low temperatures, an A wall quickly falls to the bottom of a valley only to move to the bottom of a neighboring lower valley when $\ln t$ reaches the barrier height of the intervening bond. Likewise, B walls move from top to top of "mountains." When an A and a B meet, they annihilate preserving the alternating *ABAB* sequence. Analytic treatment is difficult because decimation generates correlations among the A and B positions. Performing the RSRG numerically on a large sample [16] we find that the system evolves to a state with one A at each minimum and one B at each maximum of the renormalized landscape. We thus make the ansatz that this is the correct form of the asymptotic states. The RG analysis is then again simple [14], and the results asymptotically exact if the ansatz is correct. The equal time spin correlations can be obtained from the difference between the probabilities that an even or an odd number of extrema of the renormalized potential—i.e., domain walls—exist between a given pair of points. For a symmetric distribution of random fields, we obtain

$$\overline{\langle S_0(t) S_L(t) \rangle} \approx \sum_{n=-\infty}^{\infty} \frac{48 + 32(2n+1)^2 \pi^2 X}{(2n+1)^4 \pi^4} e^{-(2n+1)^2 \pi^2 X}$$

with $X = \frac{L}{\Gamma^2}$, distances normalized as earlier and $\Gamma = T \ln t$. At sufficiently long times $\Gamma > \Gamma_J = 4J$, we can no longer ignore creation of pairs of walls. But, at this point, the energy cannot be lowered further by *any* process. Thus if the renormalization is stopped at Γ_J, in the small field, the low T scaling limit the configuration of the walls corresponds, up to negligible thermal fluctuations, to the equilibrium state. The above equation should then give the mean equilibrium spin correlation function with lengths measured in units of the Imry-Ma length above which the random fields dominate the exchange.

Since 1D Ising spin glasses in a field are equivalent via a gauge transformation to random field ferromagnets, we can also obtain results for such a system. If a large magnetic field is quickly reduced to be $\ll J$ but nonzero, the domain wall dynamics will be like that for the ferromagnet with domain walls initially at every extremum of the potential. The decay of the magnetization for log times up to Γ_J is given by the difference between the probabilities that a spin has flipped an even or an odd number of times. We obtain $\overline{\langle S_i(t) \rangle} \sim \bar{l}(t)^{-\lambda}$ with $\lambda = \frac{1}{2}$. Note that this value of λ saturates the lower bound of $d/2$ in contrast to the pure 1D Ising case which saturates the upper bound of $\lambda = d$ [3]. For the symmetric random-field Ising model (RFIM) one similarly finds that $\overline{\langle S_i(t) S_i(t') \rangle} \sim [\bar{l}(t')/\bar{l}(t)]^{\frac{1}{2}}$.

We next study "persistence" properties. One must now carefully distinguish between the *effective dynamics* (i.e., the walker jumping between valley bottoms) and the *real* dynamics. The probability $N(t)$ that a *single walker* has *never* crossed its starting point $x(0) = x_0$ between 0 and t can be found by placing an absorbing boundary at x_0 and using methods similar to the calculation of the end point magnetization in the RTFIC [13]. We find

$N(t) \sim \bar{l}(t)^{-\theta_1}$ at large times with $\theta_1 = \frac{1}{2}$ (cf. $\theta_1 = 1$ in the pure case). A related quantity, $M(t)$, is the fraction of starting points, x_0, for which the *thermally averaged position* $\langle x(t) \,|\, x(0) = x_0 \rangle$ never crosses x_0 up to time t. While in a single "run" in a given environment the walker typically crosses its starting point many times while trapped in a valley, averaging over many runs in the same environment yields a $\langle x(t) \rangle$ which crosses x_0 exactly once each time the bond on which x_0 lies is decimated since this causes its valley bottom to cross x_0. At long times, the probability $M(t)$ thus reflects the effective dynamics, in particular, the probability that the bond on which x_0 lies has *never* been decimated before time t yielding $M(t) \sim \bar{l}(t)^{-\bar{\theta}_1}$ with $\bar{\theta}_1 = (3 - \sqrt{5})/4$. Indeed, in the biased case, $M(t)$ is like the spontaneous magnetization in the RTFIC, i.e., $M(t) \sim |\delta|^\beta$ for small δ with $\beta = (3 - \sqrt{5})/2$ [13].

More generally, in the effective dynamics, the probability of exactly n crossings of the origin up to time t scales as $\ln(\ln t)$ in the unbiased case. The rescaled variable $g = n/\ln(\ln t)$ has a multifractal distribution $\mathrm{Prob}(g) \sim \bar{l}(t)^{-\bar{\theta}_1(g)}$:

$$\bar{\theta}_1(g) = \frac{g}{2} \ln[2gs(g)] + \frac{3}{4} - \frac{s(g)}{2} \qquad (4)$$

with $s(g) = g + \sqrt{g^2 + \frac{5}{4}}$. Since $\bar{\theta}_1(\frac{1}{3}) = 0$, $g = \frac{1}{3}$ with probability 1 at large times. For a given walker, $\Xi(t) \equiv \frac{1}{t} \int_0^t x(\tau)\, d\tau$ will typically behave like $\langle x(t) \rangle$. We conjecture that the probability of $n = g \ln(\ln t)$ sign changes of Ξ up to time t decays with the same exponent $\bar{\theta}_1(g)$ for $g \le \frac{1}{3}$. For larger g, the behavior is dominated by rare valleys with almost degenerate minima on opposite sides of the origin which yield extra sign changes in $\Xi(t)$.

The persistence properties of the RFIM can similarly be analyzed. The probability $\Pi(t) \sim \bar{l}(t)^{-\theta}$ that a given *spin* at $x = 0$ has never flipped up to time t is equal to the probability that neither the nearest domain wall on one side nor the nearest (opposite type) domain wall on the other side has crossed $x = 0$. Assuming the nature of the asymptotic state is as described earlier, we find $\theta = 2\theta_1 = 1$ (cf. $\theta = \frac{3}{4}$ in the pure case [17]). In contrast, the decay of the probability that an initial *domain* survives up to time t is $S(t) \sim \bar{l}(t)^{-\psi}$ with $\psi = (3 - \sqrt{5})/4 = 0.191$ (cf. $\psi = 0.252$ in the pure case [18]).

Finally, one can study a broad class of reaction diffusion (RD) where all particles diffuse in the *same* unbiased energy landscape and react or annihilate upon meeting, for example, identical particles A which react as $A + A \to A$ with probability $1 - r$ or annihilate $A + A \to \varnothing$ with probability r. The fraction of the valleys with no particle in them tends to $p_\varnothing$, the stationary probability for the reaction process upon merging two valleys. Generalizing the absorbing boundary method we obtain that, very generally, the probability that $x = 0$ has not been crossed by *any* particle up to t decays with the exponent $\theta = 1 - p_\varnothing$, in our example, $\theta(r) = 1/(1 + r)$. The corresponding exponent $\bar{\theta}(r)$ associated with the thermally averaged particle trajectories is the solution of the hypergeometric equation [16]:

$$\bar{\theta} U(-r/(1+r), 2\bar{\theta}, 1) = U(-r/(1+r), 2\bar{\theta} + 1, 1) .$$

Remarkably, this $\bar{\theta}(r)$ is very close numerically to half the exact pure system result [17] $\frac{1}{2}\theta_{\mathrm{pure}}(r) = -\frac{1}{8} + \frac{2}{\pi^2}[\arccos(\frac{r-1}{\sqrt{2}(r+1)})]^2$.

To conclude, we have applied a RSRG method to 1D random walks in the presence of static random forces and obtained asymptotically exact results for coarsening dynamics, diffusion reaction models, and aging phenomena; surprisingly by simpler means than for the corresponding pure models. Extensions and details are given in [16].

This work has been supported in part (D. S. F.) by the NSF via Grants No. DMR 9630064 and No. DMS 9304580 and Harvard University's MRSEC.

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