Duality Transformation in a Three Dimensional Conducting Medium with Two Dimensional Heterogeneity and an In-Plane Magnetic Field

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The classical duality transformation of two dimensional continuum conductors is extended to three dimensional conductors with a two dimensional heterogeneity. This is used to discuss the classical magnetotransport of a periodic array of parallel cylindrical inclusions, which are either perfect insulators or superconductors, embedded in a free electron conducting host. A detailed understanding of the local field and current distributions is thereby achieved, and closed form expressions are obtained for the strong field magnetoresistance in some important configurations. [S0031-9007(98)05772-X]

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Duality transformations appear in many different physical contexts. One of the best known is the duality transformation in a two dimensional (planar coordinates x, y) heterogeneous conductor [1–4]. In such a system, the local electric current density $\mathbf{J}(x, y)$ and electric field $\mathbf{E}(x, y)$ are both planar, and are related locally through a local conductivity tensor $\hat{\sigma}(x, y)$ in the usual way $\mathbf{J} = \hat{\sigma} \cdot \mathbf{E}$. The transformation consists of rotating the two vector fields by 90° in the x, y plane, so as to obtain a new pair of "dual fields" $\mathbf{E}_D \equiv \mathbf{e}_z \times \mathbf{J}$ and $\mathbf{J}_D \equiv \mathbf{e}_z \times \mathbf{E}$, which are the electric and current density fields of the dual electrical conduction problem, where the local conductivity tensor $\hat{\sigma}_D$ is simply related to $\hat{\sigma}$ and the boundary conditions are appropriately related to those of the original problem.

This transformation works even when $\hat{\sigma}$ is nonsymmetric; therefore it can be applied in the presence of a perpendicular magnetic field [5]. It has some important physical consequences: The critical exponent t, which describes how the bulk effective conductivity of a metal-insulator mixture tends to 0 at a conductivity threshold, as a function of the metal volume fraction or some other structural parameter, must be equal to the critical exponent s, which describes how the bulk effective conductivity of a superconductor-normal conductor mixture with a similar microstructure tends to ∞, as a function of the same parameter [6]. Other consequences of duality are the fact that the bulk effective Hall resistivity ρ_{He} of a two component isotropic composite can be written entirely in terms of the bulk effective Ohmic resistivity ρ_e of that system [7,8] and the fact that the bulk effective magnetoresistivity of a metal-insulator mixture can be obtained from the magnetoresistivity of the pure metal and the bulk effective zero field resistivity [7,8].

This duality symmetry is based upon the fact that a two dimensional (2D) curl-free vector field becomes divergence-free when it is rotated by 90° in the plane, while a 2D divergence-free vector field becomes curl-free when it is similarly rotated. Three dimensional (3D) vector fields usually lack these properties. We will show that, for the special case of electrical conduction in a 3D system with a 2D heterogeneity, a nontrivial duality transformation does exist. We will then use that transformation in order to gain insight about the magnetotransport properties of a two component composite medium of that type. We will obtain some surprising results which will be compared with numerical computations on such systems.

Consider a heterogeneous medium, characterized by a local resistivity tensor that depends only upon y and z; i.e., the x axis is an axis of cylindrical symmetry. If the boundary conditions on the electric potential or the normal current density are macroscopically uniform, then the local values of $\mathbf{E}$ and $\mathbf{J}$ will also be independent of x, with the possible exception of regions near the boundaries. From the local curl-free condition satisfied by $\mathbf{E}$ it then follows that E_x is constant everywhere. It is easy to see that the dual electric field, defined by $\mathbf{E}_D \equiv (E_x, -\rho_{00}J_z, \rho_{00}J_y)$, is curl-free, while the dual current density, defined by $\mathbf{J}_D \equiv (J_x, -E_z/\rho_{00}, E_y/\rho_{00})$, is divergence-free. The y and z components of these 3D fields were obviously obtained by rotating the y, z components of $\mathbf{J}$ and $\mathbf{E}$ by 90°, respectively, in the y, z plane. The constant resistivity factor ρ_{00}, which remains to be specified, has been introduced so as to ensure that the components of $\mathbf{E}_D$ and $\mathbf{J}_D$ have consistent physical dimensions. These fields are related by $\mathbf{J}_D = \hat{\sigma}_D \cdot \mathbf{E}_D$, where the precise form of the "dual conductivity tensor" $\hat{\sigma}_D$ can easily be worked out and is usually nonsymmetric, even if $\hat{\sigma}$ was symmetric. Because any linear combination of $\mathbf{E}$ and $\mathbf{E}_D$ will be curl-free, and any linear combination of $\mathbf{J}$ and $\mathbf{J}_D$ will be divergence-free, therefore one may attempt to find such combinations that will be related by a symmetric conductivity tensor, as was done in Ref. [8] for a strictly 2D system. This would allow results which are available for problems with symmetric conductivity tensors to be applied to cases where $\hat{\sigma}$ is nonsymmetric.

We will restrict the subsequent discussion to systems where a periodic square array of identical, parallel, nonoverlapping inclusions of cylindrical shape, with nearest neighbor distance a and radius $R < a/2$, and with the cylindrical symmetry axis taken as the x axis, is embedded in a uniform, free electron metal host, and the entire system is subject to a strong applied magnetic field $\mathbf{B} \parallel z$. Such systems have recently begun to be studied both theoretically [9,10] and experimentally [11]. The host is characterized by a resistivity tensor of the form

$$\hat{\rho}_{\text{host}} = \rho_0 \begin{pmatrix} 1 & -H & 0 \\ H & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{1}$$

where ρ_0 is the Ohmic resistivity, which is independent of $\mathbf{B}$, and $H = \omega_c \tau = \mu |\mathbf{B}|$ denotes the Hall-to-Ohmic resistivity ratio ($\omega_c = e|\mathbf{B}|/mc$ is the cyclotron frequency and μ is the Hall mobility—both have the same sign as the charge e of the charge carriers, τ is the transport relaxation time). We will henceforth use $\rho_{00} = \rho_0$ in the definition of the dual fields—this value will, of course, also be used in the inclusions. Because $\mathbf{B}$ does not lie along the cylindrical symmetry axis, therefore the electrical transport always has a 3D character: Even if the volume averaged current density $\langle \mathbf{J} \rangle$ lies in the y, z plane, the local Hall effect in the host, in conjunction with the heterogeneous microstructure, will usually induce local fields and currents in all directions.

The dual conductivity tensor of the host, obtained from the resistivity tensor of (1), is symmetric,

$$\hat{\sigma}_{D\,\text{host}} = \frac{1}{\rho_0} \begin{pmatrix} 1 & 0 & H \\ 0 & 1 & 0 \\ H & 0 & 1 + H^2 \end{pmatrix}. \tag{2}$$

When $|H| \gg 1$ this represents an extremely anisotropic conductor, with principal axes that are only slightly rotated about the y-coordinate axis, and with a very large conductivity in the z direction. If the inclusions are *perfectly insulating*, then their dual conductivity tensor has the form

$$\hat{\sigma}_{D\,\text{inc}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \infty & 0 \\ 0 & 0 & \infty \end{pmatrix}. \tag{3}$$

We now use the dual problem in order to study the magnetotransport in such a composite medium for the limit of a very strong applied magnetic field, namely, $|H| \gg 1$.

If the array of inclusions is aligned so that its principal axes are along y and z, and if we apply a boundary condition such that $\langle \mathbf{J} \rangle = \mathbf{e}_z \parallel \mathbf{B}$, then, because $\mathbf{J} = 0$ in the inclusions and $E_x = \rho_0(J_x - HJ_y)$ in the host, therefore, integrating the last equation over the host subvolume, we conclude that $E_x = \langle E_x \rangle = 0$. The boundary condition for the dual problem is therefore such that $\langle \mathbf{E}_D \rangle = -\rho_0 \mathbf{e}_y$. Because dual current must sometimes flow through the host in the y direction, but it costs very little to flow through the host in the z direction, and it costs nothing at all to flow through the obstacles (since $\sigma_{D\,\text{inc}} = \infty$ in those directions), therefore we expect a flow pattern in the dual problem as shown qualitatively in Fig. 1(a). This translates into a flow pattern for the original problem as shown in Fig. 1(b): $J_x = J_y = 0$ everywhere, and $J_z = 0$ in the slab-shaped regions defined by the rows of parallel obstacles. In between those slabs, J_z must then have the uniform value

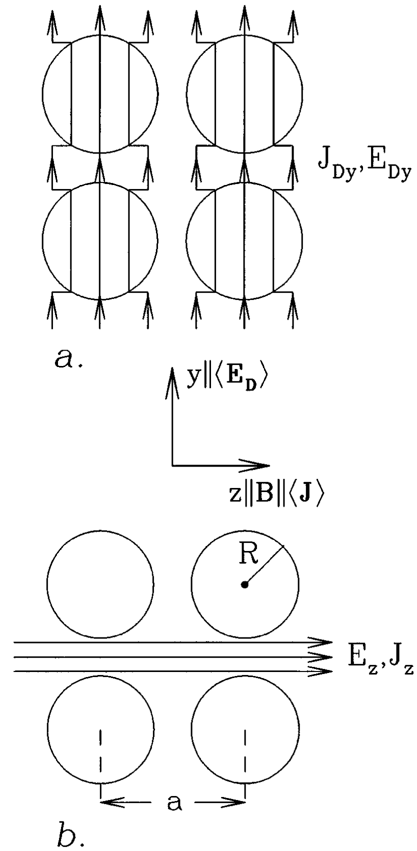

FIG. 1. (a) Distribution of dual currents in longitudinal transport along (001) axis of insulating cylinder array when $|H| \gg 1$. J_{Dy} is uniform in the obstacle-free slabs perpendicular to y in the host. (b) Distribution of physical current: $J_z \neq 0$ only in the above-mentioned slabs. The other components of $\mathbf{J}$ vanish everywhere.

$$J_z = \frac{\langle J_z \rangle}{1 - 2R/a}. \tag{4}$$

In order to test this surprising prediction we computed the detailed current distribution for such a system, using a method developed earlier [9,12]. The results essentially agree with the above expectations—see Fig. 2: The hot spots at the inclusion edges are probably due to the squeezing of J_z at those points—only at those z values does J_z strictly vanish outside the unobstructed slabs when $|H| < \infty$. These results imply that the distribution of $\mathbf{J}(\mathbf{r})$ is saturated for large $\mathbf{B}$, and therefore that the local dissipation rate $W = \rho_0 \mathbf{J}^2$ is saturated as well, and along with it the bulk effective longitudinal resistivity $\rho_{\parallel}^{(e)}$. This agrees both qualitatively and quantitatively with results of earlier computations [see Figs. 12(b), 12(e), and 12(h) of Ref. [9]]. A similar behavior occurs whenever $\langle \mathbf{J} \rangle \parallel \mathbf{B}$ lies along a low order lattice axis, and the obstacles are

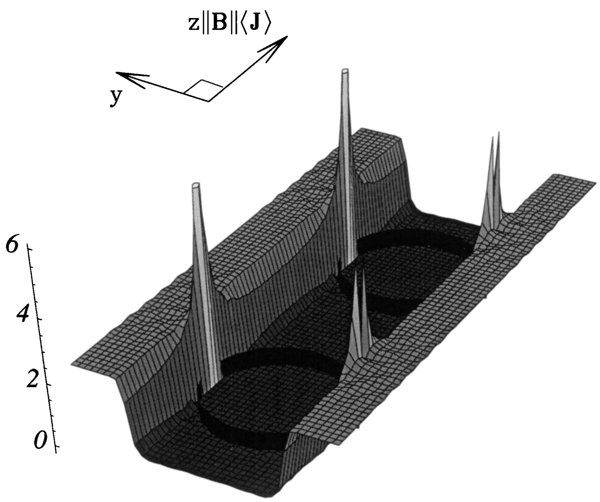

FIG. 2. Numerically computed 3D plot of $J_z(y,z)$ for longitudinal transport along (001), using insulating cylinders with $R/a = 0.4$ and $H = 20$.

small enough so that slab-shaped regions parallel to $\langle \mathbf{J} \rangle$ exist that are obstacle-free, as in Fig. 1(b). For other directions of $\mathbf{B}$, the current lines cannot be straight, even at large $\mathbf{B}$—they must distort in order to circumvent the obstacles. Thus, there will be nonzero local values of J_y, and consequently also of J_x, in most of the host subvolume. Furthermore, even though J_y and J_z may saturate for large $\mathbf{B}$, J_x will never saturate, because $J_x = HJ_y$. Therefore, W will not saturate for most directions of $\mathbf{B}$, and consequently $\rho_{\parallel}^{(e)}$ must exhibit strong oscillations with the direction of $\mathbf{B}$. That, too, is in agreement with previous computations [9].

Continuing to align the principal axes of the square array of obstacles along the y and z axes, we now apply a boundary condition such that $\langle \mathbf{J} \rangle = \mathbf{e}_y \perp \mathbf{B}$. In that case, E_x no longer vanishes, but has the constant value (p_{host} is the volume fraction of the host subvolume)

$$E_x = -H\rho_0 \langle J_y \rangle / p_{\text{host}} = -H\rho_0 / p_{\text{host}} . \tag{5}$$

The dual problem then satisfies $E_{Dx} = -H\rho_0/p_{\text{host}}$, $\langle E_{Dy} \rangle = 0$, $\langle E_{Dz} \rangle = \rho_0$. In the limit $|H| \gg 1$, the dual current lines in the y,z plane will be straight lines in the z direction, but the magnitude of J_{Dz} will depend upon y in such a way that the total voltage drop along all current lines is the same. Using this picture, we can easily calculate $E_{Dz}(y)$ in the host, and consequently $J_y(y)$ in the host (here y is measured with respect to the cylinder axis in one unit cell),

$$J_y = -\frac{E_{Dz}}{\rho_0} = \begin{cases} (1 - \frac{2}{a}\sqrt{R^2 - y^2})^{-1}, & |y| < R, \\ 1, & R < |y| < a/2. \end{cases} \tag{6}$$

In this situation $J_z = E_{Dy}/\rho_0$ is negligible; however, $J_x = J_{Dx} = (E_{Dx} + HE_{Dz})/\rho_0$ has very large values, of order H, even though its average value vanishes. That is why the dissipation rate does not saturate. Its volume average gives the bulk effective in-plane transverse resistivity, which is easily found for large fields from (6),

$$\frac{\tilde{\rho}_{\perp}^{(e)}}{\rho_0} \cong H^2 \bigg[1 - 2\frac{R}{a} - \frac{1}{1 - \pi(R/a)^2} - \frac{\pi}{2} + \frac{2}{\sqrt{1 - 4R^2/a^2}} \arctan\bigg(\frac{1 + 2R/a}{1 - 2R/a} \bigg)^{1/2} \bigg]. \tag{7}$$

For $R/a = 0.4$ this leads to $\tilde{\rho}_{\perp}^{(e)}/\rho_0 \cong 0.782H^2$, in good agreement with Fig. 12(c) of Ref. [9]. In Fig. 3 we show a numerically computed 3D plot of $J_y(y,z)$ for this case, when $H = 20$, which agrees quantitatively with (6).

If, instead of aligning $\mathbf{B}$ and $\langle \mathbf{J} \rangle$ along the principal axes (001) and (010), we align them along the 45° lattice axes (011), $(01\bar{1})$, then similar considerations lead to a more complicated, but still closed form, expression for $J_y(y)$. The asymptotic behavior of $\tilde{\rho}_{\perp}^{(e)}$ at strong fields when $R/a = 0.4$ is then given by $\tilde{\rho}_{\perp}^{(e)}/\rho_0 \cong 0.0240H^2$. Although this means that $\tilde{\rho}_{\perp}^{(e)}$ does not saturate when $\mathbf{B} \parallel (011)$, the coefficient of the H^2 term is much smaller than when $\mathbf{B} \parallel (001)$. Therefore a very strong angular dependence of $\tilde{\rho}_{\perp}^{(e)}$ on the direction of $\mathbf{B}$ will be observed. We note that earlier numerical computations, though consistent with these results, were not sufficiently precise to detect the small H^2 term in $\tilde{\rho}_{\perp}^{(e)}$ for $\mathbf{B} \parallel (011)$ in this sample—see Fig. 12(c) in Ref. [9]. Only the present discussion is able to assert that, even for $\mathbf{B}$ in this direction, $\tilde{\rho}_{\perp}^{(e)}$ does not saturate but includes an H^2 term with a small but calculable coefficient.

Finally, instead of inclusions that are *perfect insulators*, we now consider the other extreme situation, namely, a square array of *superconducting cylindrical inclusions,*

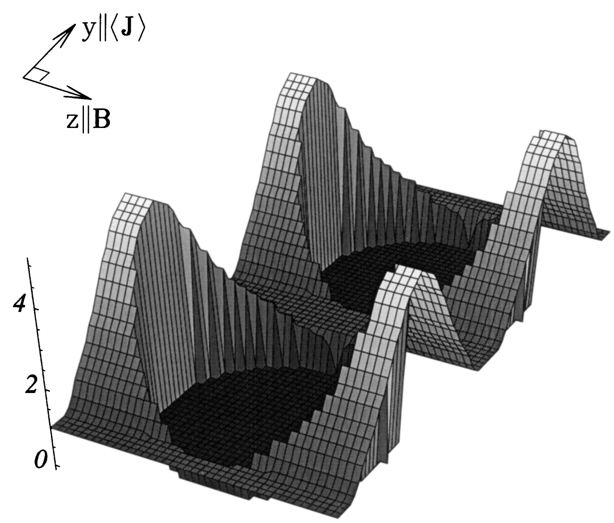

FIG. 3. Numerically computed 3D plot of $J_y(y,z)$ for transverse transport along (010), using insulating cylinders with $R/a = 0.4$ and $H = 20$.

again embedded in a free electron host. In that case, Eq. (3) is replaced by

$$\hat{\sigma}_{D\,\mathrm{inc}} = \begin{pmatrix} \infty & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{8}$$

Because the superconducting cylinders span the entire thickness of the film, therefore $E_x = 0$ everywhere. By contrast, although we will usually assume boundary conditions such that $\langle J_x \rangle = 0$, locally J_x will usually not vanish: Because $E_x = \rho_0(J_x - HJ_y)$ in the host, therefore we will have there $J_x = HJ_y$. Thus, even if J_y and J_z saturate as $H \to \infty$, J_x and W will not saturate unless $J_y = 0$ everywhere.

For transverse transport boundary conditions and $|H| \gg 1$, when $\langle \mathbf{J} \rangle = \mathbf{e}_y \perp \mathbf{B}$ and both $\mathbf{B}$ and $\langle \mathbf{J} \rangle$ lie along principal axes of the square array, the dual problem results in J_{Dz}, E_{Dz} being nonzero and uniform in the unobstructed slab-shaped regions and zero elsewhere, like J_z, E_z in Fig. 1(b), and $J_{Dy} = J_{Dx} = 0$ everywhere. Therefore, in the original problem, J_y will be uniform in the unobstructed slabs, and will flow entirely through the superconducting cylinders in the slabs defined by them, like J_{Dy} in Fig. 1(a). It then follows that the dissipation, and along with it $\tilde{\rho}_{\perp}^{(e)}$, do not saturate with H, and we can easily calculate their leading behavior,

$$\tilde{\rho}_{\perp}^{(e)}/\rho_0 \cong H^2(1 - 2R/a)\,. \tag{9}$$

That value will be a maximum in the angular plot of $\tilde{\rho}_{\perp}^{(e)}$ vs the direction of $\mathbf{B}$. Similar maxima can be expected to appear along any lattice axis that features perpendicular slabs of pure, unobstructed host material. It is interesting to note that the coefficient of H^2 in (9) grows as the cylinder radii decrease. At some point this tendency must change since, for $R = 0$, there is no dissipation at all, and furthermore E_x does not vanish. Clearly, the behavior of such systems when R/a becomes very small deserves further study.

For a superconducting cylinder array under longitudinal transport boundary conditions, with $\langle \mathbf{J} \rangle \parallel \mathbf{B} \parallel (001)$, the dual problem has the property that in the host it is cheapest for the current to flow in the z direction. Therefore J_{Dy} will be independent of z in the host for any value of y, but it will vanish inside the inclusions. E_{Dy} will exhibit similar behavior in the host, while $E_{Dz} \ll E_{Dy}$ and $E_{Dx} \ll E_{Dy}$. Therefore J_z will be independent of z for any value of y in the host, and the other components of $\mathbf{J}$ are negligible; i.e., the current lines will be straight lines in the z direction. The asymptotic strong field shape of $J_z(y)$ can be computed in closed form by first computing J_{Dy}, which is given by the same expression as J_y of (6). The longitudinal magnetoresistance in the (001) direction will therefore saturate for large H, and its value will be given by

$$\frac{\rho_{\parallel}^{(e)}}{\rho_0} \cong \left[1 - 2\frac{R}{a} - \frac{\pi}{2} + \frac{2}{\sqrt{1 - 4R^2/a^2}} \times \arctan\left(\frac{1 + 2R/a}{1 - 2R/a}\right)^{1/2}\right]^{-1}. \tag{10}$$

In summary, we have extended the classical duality transformation to a 3D classical continuum conductor which exhibits a 2D heterogeneity. For composite films made of a free electron conducting host with a periodic array of cylindrical inclusions that are either superconducting or perfectly insulating, we used this transformation to discuss magnetotransport when the magnetic field lies in the film plane. When that field is strong, the current distributions and angular dependence of the magnetoresistance have a surprisingly rich behavior. Much insight, as well as quantitative results for asymptotic behavior at strong fields, were obtained in this way.

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