Microwave Realization of the Hofstadter Butterfly

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The transmission of microwaves through an array of 100 scatterers inserted into a waveguide is studied. The length of each scatterer could be varied individually, thus allowing the realization of arbitrary scattering arrangements. For periodic sequences with varying period length the found transmission bands reproduced the Hofstadter butterfly, originally predicted for the spectra of conduction electrons in strong magnetic fields. In the experiment it was used that the same transfer matrix formalism is applicable to both the microwave and the electronic systems. [S0031-9007(98)05781-0]

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In 1976 Hofstadter [1] published a work on the spectra of Bloch electrons on a two-dimensional lattice with lattice constant a, and a perpendicularly applied uniform magnetic field B. The spectra contain one single parameter $\alpha = a^2eB/h$ counting the number of magnetic flux quanta per unit cell.

In the tight binding approximation the Schrödinger equation of the system reduces to the Harper equation

$$\psi_{n+1} + 2\cos(2\pi n\alpha - \alpha_0)\psi_n + \psi_{n-1} = E\psi_n\,, \quad (1)$$

where ψ_n is the wave function at site n, and α_0 is a phase associated with the linear momentum of the electron, superimposed on the cyclotron orbits in the magnetic field. E is the energy in normalized units. All possible eigenenergies of the Harper equation are in the range $-4 \le E \le 4$, where α may be restricted to the range $0 \le \alpha \le 1$. In fact, it is sufficient to consider the range $0 \le \alpha \le 1/2$, as the Harper equation is invariant under the substitution $\alpha \to 1 - \alpha$. The Harper equation (1) can alternatively be written in the form

$$\begin{pmatrix} \psi_{n+1} \\ \psi_n \end{pmatrix} = T_n \begin{pmatrix} \psi_n \\ \psi_{n-1} \end{pmatrix}, \quad (2)$$

with the transfer matrix

$$T_n = \begin{pmatrix} E - 2\cos(2\pi n\alpha - \alpha_0) & -1 \\ 1 & 0 \end{pmatrix}. \quad (3)$$

The set of all eigenvalues of the Harper equation if plotted in the (E, α) plane form the Hofstadter butterfly (see Fig. 1 of Ref. [1]).

The unceasing interest in the system from the very beginning has its cause in the self-similar structure of the butterfly. For rational values $\alpha = p/q$ the magnetic unit cell is by a factor q larger than the lattice unit cell causing a splitting of all electronic Bloch bands into q subbands. For irrational values of α the spectra become fractal and form a Cantor set, resulting in unusual level spacing distributions and diffusion properties [2] (for a review see Ref. [3]).

For an experimental realization of the butterfly with typical lattice spacings of some 0.1 nm magnetic fields of about 10^5 T are necessary, which is far beyond the technically accessible limit. The only way to circumvent this problem is to use artificial superlattices, and, in fact, here first indications of a magnetic induced subband splitting due to commensurability effects have been found [4,5]. Another possible system is one of Wigner crystals formed by the crystallization of a two-dimensional electron gas in a strong magnetic field under the influence of the Coulomb repulsion. As the lattice constant of the Wigner crystal depends on the filling factor of the band, here, too, commensurability effects are expected and, indeed, observed [6]. In the mentioned works the splitting of an electron band into two subbands is reported, but measurements over a larger α range were not performed which is indispensable for an unambiguous identification of the butterfly.

In the present paper a completely different idea is used, which yields the butterfly over the complete α and energy range. The approach is based on an analogy between electronic and photonic systems, where concepts originally developed in solid state physics, such as band structures, and localization, are applied to the propagation of the electromagnetic waves through periodic and random systems [7]. Microwave experiments showing photonic band gaps in dielectric lattices have already been performed by Yablonovitch and Gmitter [8] as well as by McCall *et al.* [9]. Krug [10] and later Prange and Fishman [11] proposed to study kicked quantum systems via the propagation of light through optical fibers with modulated refraction indices.

In the present Letter, a nearby equivalence of the Harper equation, and an equation describing the propagation of waves (in our case microwaves) in a one-dimensional array of scatterers is developed. Essentially the same transfer matrix technique as for the Harper equation can also be used to describe the propagation of microwaves through a scattering array. In a one-dimensional waveguide with only one possible mode the amplitudes a_n, b_n of the waves propagating to the left and to the right, respectively, are obtained from

$$\begin{pmatrix} a_{n+1} \\ b_{n+1} \end{pmatrix} = T_n \begin{pmatrix} a_n \\ b_n \end{pmatrix}, \quad (4)$$

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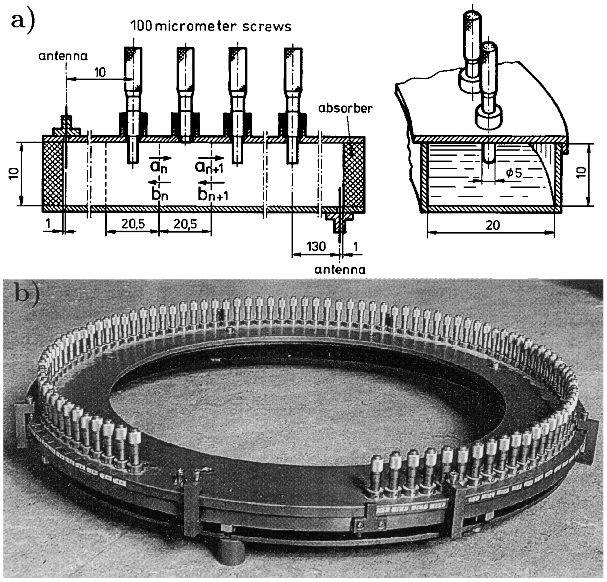

FIG. 1. (a) Schematic view of the waveguide ($a = 20$ mm, $b = 10$ mm). The microwaves are coupled in through the left antenna, and the transmission through the waveguide with 100 scatterers (micrometer screws) is measured with the right antenna. At each end the waveguide is closed by microwave absorbers. (b) Photograph of the apparatus.

where T_n is the transfer matrix associated with the nth scatterer [see Fig. 1(a)]. From time-reversal symmetry follows that the transfer matrix can be written as

$$T_n = \begin{pmatrix} \frac{1}{|t_n|} e^{\imath(\theta+\gamma_n)} & \imath \frac{|r_n|}{|t_n|} e^{-\imath\theta} \\ -\imath \frac{|r_n|}{|t_n|} e^{\imath\theta} & \frac{1}{|t_n|} e^{-\imath(\theta+\gamma_n)} \end{pmatrix}, \tag{5}$$

where $|t_n|$, $|r_n|$ are the moduli of transmission, and reflection amplitudes, obeying $|t_n|^2 + |r_n|^2 = 1$. γ_n is the phase of the transmission amplitude. $\theta = kd/2\pi$ is the phase shift from the free propagation between the scatterers, where d is the distance between the scatterers, and k the wave number.

For the case $|\mathrm{Tr}(T_n)| < 2$ both transfer matrices (5) and (3) can be written in the form

$$T_n = e^{\imath \phi_n \sigma_n}, \tag{6}$$

where

$$\cos \phi_n = \frac{\cos(\theta + \gamma_n)}{|t_n|}, \tag{7}$$

$$\sigma_n = \frac{1}{\sin \phi_n} \begin{pmatrix} \frac{\sin(\theta+\gamma_n)}{|t_n|} & \frac{|r_n|}{|t_n|} e^{-\imath\theta} \\ -\frac{|r_n|}{|t_n|} e^{\imath\theta} & -\frac{\sin(\theta+\gamma_n)}{|t_n|} \end{pmatrix}, \tag{8}$$

for the scattering system and

$$\cos \phi_n = \frac{E}{2} - \cos(2\pi n\alpha - \alpha_0), \tag{9}$$

$$\sigma_n = \frac{1}{\imath \sin \phi_n} \begin{pmatrix} \cos \phi_n & -1 \\ 1 & \cos \phi_n \end{pmatrix}, \tag{10}$$

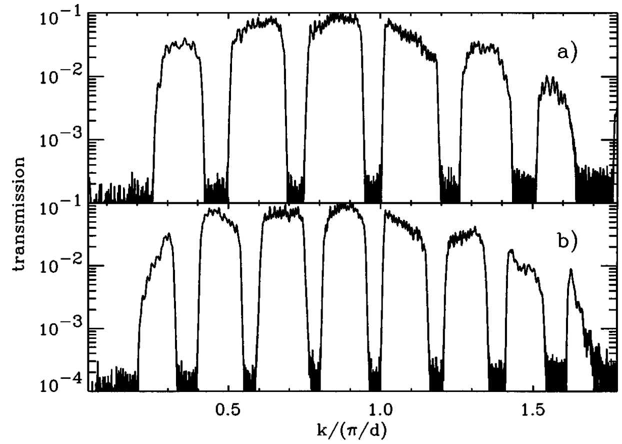

FIG. 2. Transmission probability for a periodic scatterer arrangement showing the forbidden and allowed Bloch bands. In (a) every third and in (b) every fourth scatterer was introduced 3 mm. The shown wave number range corresponds to the frequency range from 7.5 to 15 GHz.

for the Harper equation. Because of $(\sigma_n)^2 = 1$ the eigenvalues of σ_n are $+1$ and -1. It is, hence, possible to transform σ_n to the spin matrix σ_z.

For the case $|\mathrm{Tr}(T_n)| < 2$ the ψ_n along the chain of sites are therefore obtained by successive spinor rotations. If all T_n are equal, which is the case for $\alpha = 0$, all rotations are about the same axis, and only the phase of the wave function changes from site to site.

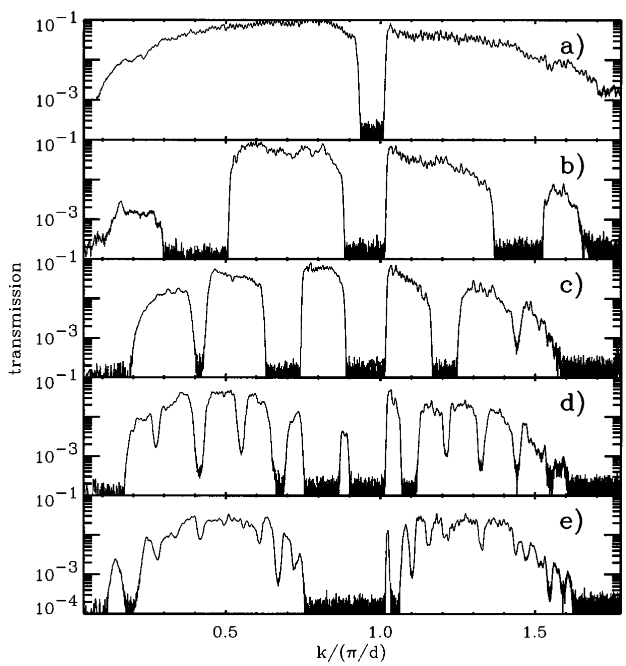

FIG. 3. Transmission spectra for different periodic arrangements with $\alpha = 1/q$, where $q = 1$ (a), 2 (b), 4 (c), 8 (d), and 16 (e). In (a) the two main Bloch bands can be seen, each of which splits into $q = 2$ subbands in (b). For larger values of q further band splitting is observed, but due to the strong absorption in the system not all q subbands can be seen.

This is the range of the allowed electronic bands. For the case $|\mathrm{Tr}(T_n)| > 2$, on the other hand, ϕ_n becomes imaginary, and the wave functions are exponentially damped. The same argumentation can be applied to the case of rational α values p/q. Now the sequence $\{T_n\}$ of transfer matrices is periodic with a period length q. In this case the value of the trace of the product $\prod_{n=1}^{q} T_n$ discriminates between the ranges of allowed and forbidden bands.

It is this equivalence between the Harper equations (1) and (2) and Eq. (4) describing the propagation of microwaves through a one dimensional array of scatterers that is the basis for the microwave realization of the Hofstadter butterfly. The forms of the σ_n are different in the two cases, but this changes only the quantitative behavior. The qualitative appearance of the spectra, the number of subbands, fractality, etc., is independent of the exact form of σ_n.

In the experiment a rectangular waveguide with dimensions $a = 20$ mm, $b = 10$ mm was used. 100 cylindrical scatterers with a radius of $r = 2.5$ mm, and a distance of $d = 20.5$ mm, could be introduced into the waveguide (see Fig. 1). The lengths of the scatterers could be varied with the help of micrometer screws. The upper part of the waveguide could be rotated against the lower one, thus allowing one to vary the position of the exit antenna, a feature which, however, was not yet used. The total transmission through the empty waveguide (with all scatterers removed) amounted to only about 15%. This rather high loss is partly due to a mismatch of the antennas, and partly due to the absorbers at the end of the waveguide. If these effects are considered, one obtains for the real transmission through the waveguide a value of about 70%. In the figures below the uncorrected transmission probabilities are displayed. We measured in the frequency range where only the first mode can propagate, starting from the cutoff frequency of $\nu_{\min} = \frac{c}{2a} \approx 7.5$ GHz up to $\nu_{\max} = \frac{c}{a} = \frac{c}{2b} \approx 15$ GHz, where the propagation of the second mode becomes possible. In a microwave waveguide the dispersion relation is given by $k = \frac{2\pi}{c}\sqrt{\nu^2 - \nu_{\min}^2}$. To avoid a distortion of the spectra for low frequencies, for all data presented the wave number k is used as abscissa

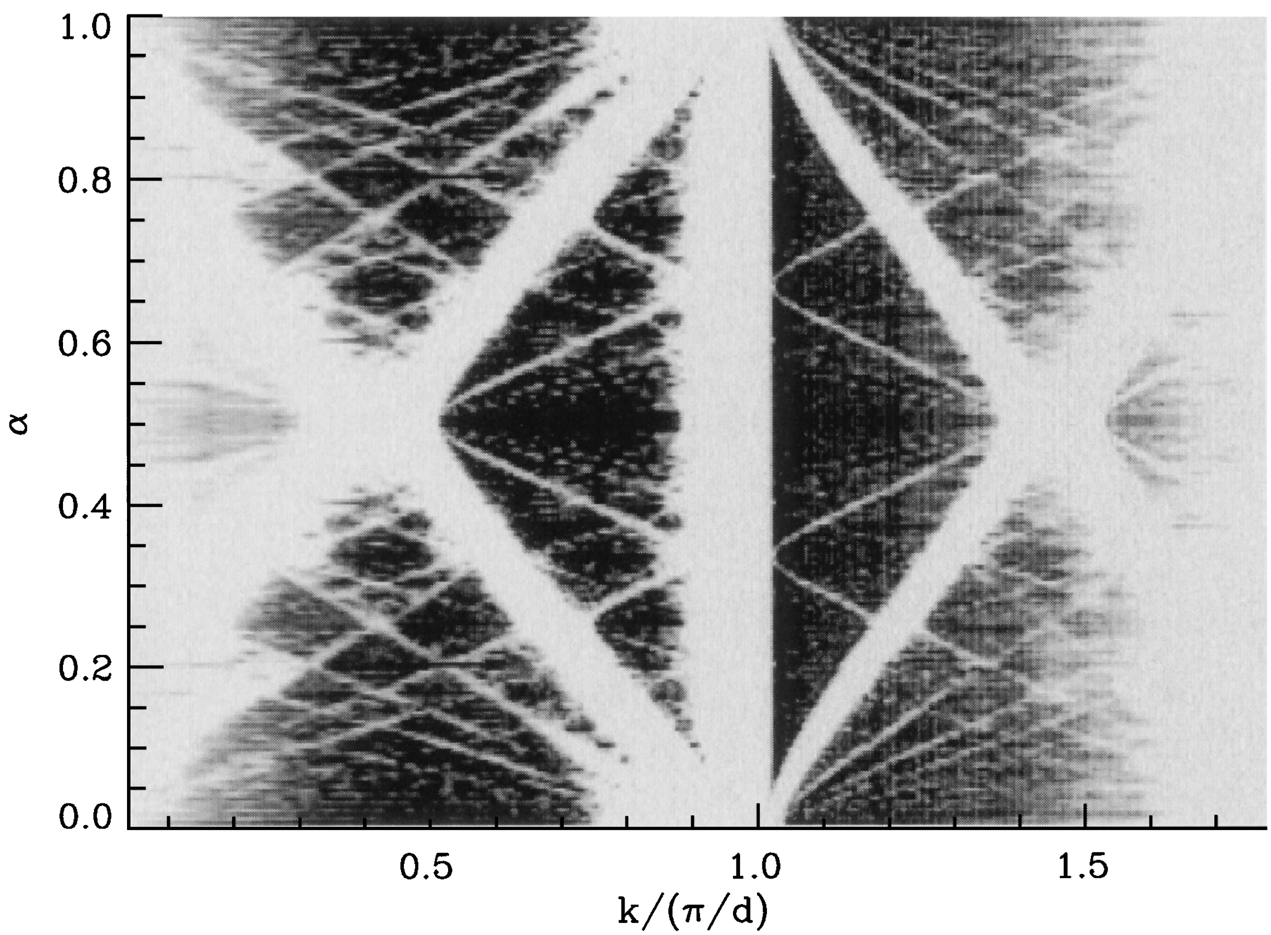

FIG. 4. Transmission spectra for a periodic arrangement of scatterers with α ranging from 0 to 1 in steps of 0.005. The upper part was obtained by reflection. All 100 scatterers were used. The first two Bloch bands are seen, showing two copies of the Hofstadter butterfly. The spectra were converted to a grey scale, where black and white corresponds to high and low transmission, respectively.

in units of π/d, where d is the distance between the scatterers.

To test the apparatus we started with periodic scattering arrangements. As an example Fig. 2 shows two transmission measurements demonstrating clearly the allowed and forbidden transmission bands.

In the context of the above-mentioned analogy one may look upon the waveguide with a periodic arrangement of scatterers as a one-dimensional photonic crystal [8].

For the realization of the butterfly a periodic modulation of the lengths of the scatterers was applied with the period length as a parameter. We did not vary, however, the lengths according to $l_n = l_0 \cos(2\pi n\alpha - \alpha_0)$ but replaced the cosinusoidal variation by a rectangular one,

$$l_n = \begin{cases} 0 & \text{for } \cos(2\pi n\alpha - \alpha_0) \leq 0\,, \\ l_0 & \text{for } \cos(2\pi n\alpha - \alpha_0) > 0\,, \end{cases}$$

which was much easier to realize. In the experiment we chose $l_0 = 3$ mm and $\alpha_0 = 0$.

Figure 3 shows a selection of transmission spectra for different $\alpha = 1/q$ values. In 3(a) the first two fundamental transmission bands are seen, corresponding to the first two Brillouin zones of the photonic crystal. In 3(b)–3(e) one finds the expected splitting into subbands, although because of the increasing absorption in the lower and upper parts of the frequency range not all expected q subbands are observed.

The spectra are easier to interpret if the transmission probabilities are converted to a grey scale, and the spectra for different values of α are plotted together. This has been done in Fig. 4. Now the structure of the Hofstadter butterfly is identified beyond all doubt, though the low and the high wave number regions are disturbed by absorption. Both two fundamental Bloch bands show a crosslike subband splitting with α. This cross is the dominant structure of the Hofstadter butterfly (see Fig. 1 of Ref. [1]). The new subbands again split crosslike. Self-similar structures are observed up to a depth of about 3.

In Fig. 5 spectra are shown where only every second scatterer was used. Therefore now four Bloch bands are seen. Since only 50 scatterers were used, now less fractal details are seen as in Fig. 4, but instead the Hofstadter butterflies in the two middle Bloch bands are complete, and only the two outer bands are again disturbed by absorption.

As the experiment allows an easy realization of arbitrary scattering arrangements, questions of transmission and localization in random or pseudorandom [12] structures may be studied equally well with the same apparatus. The possibility to also measure the field distribution

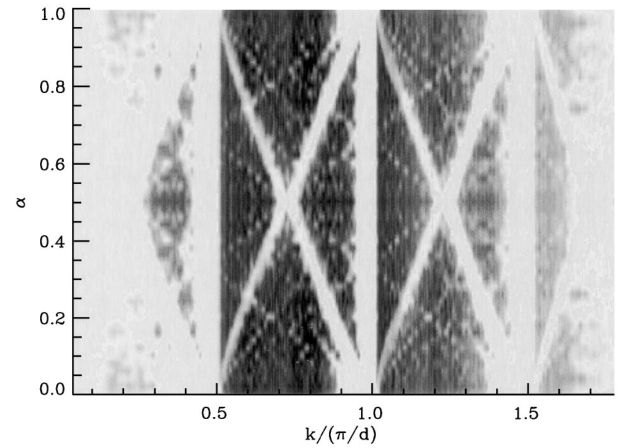

FIG. 5. As Fig. 4, but now α was varied in steps of 0.02, and only every second scatterer was used. Four Bloch bands are seen, each showing the Hofstadter butterfly.

along the waveguide is an additional interesting feature. Our doubts in the beginning that the absorption could prevent meaningful results fortunately showed up to be unfounded.

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