Synchronizing High-Dimensional Chaotic Optical Ring Dynamics

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We study chaotic ring laser systems as possible elements in a communications system. To be useful it must be possible to synchronize the transmitter and receiver lasers. We show that chaotic ring lasers can be synchronized using direct light injection from one laser into the optical cavity of the second. This synchronization occurs even when both lasers are quite high dimensional and each possesses many positive Lyapunov exponents. When the lasers are synchronized, the transmitted light can be modulated with information bearing signals and the message accurately recovered at the receiver. [S0031-9007(98)05715-9]

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The use of erbium doped ring lasers in communications devices [1] led us to inquire whether such lasers could serve as the basis for optical communications while operating in a chaotic regime. Our work focuses on a simplified prototype of the actual erbium ring laser. The delay differential equations of our model arise by integrating the partial differential equations for the electric field in the ring, so the source of high-dimensional dynamics is identified physically. We ask whether such optical systems can synchronize and support modulation and demodulation of information bearing messages. Applications aside, we demonstrate synchronization of high-dimensional dynamical systems with many positive Lyapunov exponents. Our ring lasers operate in a regime where the dimension ≥ 20, using physically natural coupling to synchronize. The only high-dimensional systems previously known to synchronize were constructed of cascaded low-dimensional subsystems which successively synchronize [2].

There have been earlier discussions of synchronization of chaotic lasers [3], also previous modeling of erbium ring lasers using delay differential equations [4]. Reference [5] discusses communicating using chaotic lasers, but the method is quite different from ours. Our method is not restricted to small amplitude messages, and in [5] output from the transmitter is modulated. We modulate the message signal into the transmitter dynamics then stability properties of the receiver assures accurate message recovery. Recent results from VanWiggeren and Roy [6] experimentally confirm many of the effects discussed here. This includes high-dimensional dynamics and the ability to modulate and demodulate messages using the methods suggested here.

To augment the numerical evidence for synchronization, we analyze the system's largest conditional Lyapunov exponent (CLE) [7]. This linear criterion is useful and powerful as long as unstable sets are not close to the synchronization manifold. The boundary determined by direct simulation is near that predicted by the exponents. Also, the lasers synchronize for a broad range of coupling parameters, improving the chance of success for a practical realization.

The literature discusses using chaotic communications for secure or encrypted signaling. We do not address this issue. There is a distinction between such communication as a means of using channel capacity and any additional potential for security. Wide-band signaling in optical systems is interesting, independent of whether the methods are cryptographically secure. Some strategies for secure communications based on low-dimensional dynamics appear susceptible to algorithmic time-series attacks [8]. Short also argues that systems with many positive Lyapunov exponents, such as our ring laser, might resist these schemes. We have not investigated our method cryptographically.

We study a "laser" which has no polarization (important in the erbium system), no imperfect passive medium, such as an optical fiber, making up the ring, and simplified atomic physics of the active medium. Power comes from externally injected light at a frequency ω_I, possibly offset from the optical frequency ω_0 of the lasing line, and through pumping of population inversion. The basic model was introduced by Ikeda some years ago [9], thus we call the system the Ikeda Ring Laser (IRL).

Our model IRL consists of four mirrors with finite reflectivity arranged at the corners of a square. (See Fig. 1). Light with electric field $E_I e^{i(\omega_I - \omega_0)t}$ (in the coordinate system rotating at ω_0) enters the ring via the upper left mirror and passes through an idealized two-level atomic system. On departing the active medium, the light reflects off four mirrors with net reflectivity B and reenters the active medium. The external injection and external pump add power, and $B < 1$ attenuates it. After a straightforward derivation we arrive at a discrete time map for the complex electric field amplitude $\zeta(t)$ at a fixed spatial location coupled to a differential equation for the spatially averaged population inversion $w(t)$:

$$\zeta(t + \tau_R) = E_I e^{i(\omega_I - \omega_0)t} + B e^{i\kappa} \zeta(t) e^{(\beta + i\alpha)w(t)}$$

$$\frac{dw(t)}{dt} = Q - 2\gamma\{w(t) + 1 + |\zeta(t)|^2 (e^{Gw(t)} - 1)/G\},$$

with τ_R the propagation time around the ring. β and α are nonlinear gain coefficients and γ the atomic decay

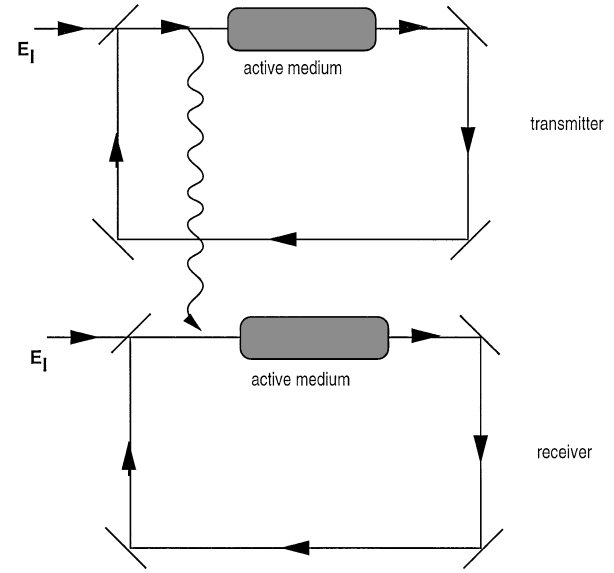

FIG. 1. Schematic of unidirectionally coupled Ikeda ring laser systems.

rate related to transitions between lasing photons and an external loss bath, G is another gain-related parameter. κ is the phase acquired by a plane wave traversing the ring. Measuring time in units of τ_R we see that only the dimensionless quantity $\gamma\tau_R$ is important in the dynamics. Physically we cannot vary γ, but we can change τ_R by varying the length of the cavity. Our time scale chooses $\tau_R = 1$. Physically accessible time scale variations are expressed as changes in γ.

This is a delay differential system. Such systems may exhibit both low-dimensional and very high-dimensional behavior: complexity generally increases with the delay time. We integrated with a simple fixed time step, δt being chosen so that $\gamma\delta t = 10^{-3}$. The value for $w(t)$ is advanced one δt using a standard fifth order Adams predictor-corrector scheme, as dw/dt explicitly depends only on $w(t)$ and $|\zeta(t)|^2$. At t the value for $\zeta(t + 1)$ is computed from the map and saved for future use.

We chose parameter values corresponding to the standard "Ikeda map," often used as an example of a two degrees of freedom chaotic system [10]. We recover the Ikeda map in the limit $\gamma \rightarrow \infty$, $G \rightarrow 0$, $\omega_I = \omega_0$, and $dw/dt = 0$,

$$\zeta(t + 1) = E_I e^{i(\omega_I - \omega_0)t} + Be^{i\kappa} e^{(-\beta - i\alpha)/(1+|\zeta(t)|^2)} \zeta(t).$$

Identifying our parameters with canonical values [10] yields $E_I = 1$, $B = 0.9$, $\kappa = 0.4$, $\beta = 0, \alpha = 6$, and $\omega_I = \omega_0$. From a related experiment [4] we estimate $G = 0.01$, and chose $\gamma = 1$ and $Q = 0$ for the calculations reported here. When $\omega_I \neq \omega_0$, the dynamical structure can be more complex than we report here, and similarly when $Q > 0$. We will report on these situations in a larger report of this work. In the matters of synchronization and modulation for information transmission, though, there is no change from the behavior reported in this short note. We have investigated a wide range of values of γ, and for γ larger than order 0.5, the dynamics are chaotic, and the essential features are the same as reported here for $\gamma = 1$. The time-asymptotic dynamics has sufficiently high dimension that the standard time-series analysis tools used for low-dimensional chaos, such as correlation dimension, false nearest neighbors, etc. [10], fail to give useful results. Specifically, false nearest neighbors calculations did not show low dimensionality with any embedding dimension up to 12, beyond which we fail to have confidence in the algorithm.

We can, however, evaluate the entire spectrum of Lyapunov exponents directly from the equations and use them to estimate the dimension [11]. Specifically, we find the exponents of the predictor-corrector integration algorithm. This is a map taking the state vector $\mathbf{S}(j)$ of the system to the state one time step δt later: $\mathbf{S}(j) \longrightarrow \mathbf{S}(j + 1) = M[\mathbf{S}(j)]$. $\mathbf{S}(j)$ is large: $2N + 5$ components for a time interval of $N = \tau_R/(\delta t)$ parts. $2N$ are from the saved values of ζ, one for $w(t)$, and four for recent derivatives of w used in the integration. We find the Jacobian $DM(\mathbf{S}[j])$ of the map along the trajectory of the system simultaneously generated by the integration algorithm. The Jacobian is too large to represent efficiently as an explicit matrix in the computation, but its sparseness allows us to write a subroutine to multiply an arbitrary vector by $DM(\mathbf{S}[j])$, and thus, any matrix, by operating on columns independently.

The Lyapunov exponents are evaluated by a variant of the standard recursive QR algorithm [10]. Given an initial random orthogonal matrix $\mathbf{Q}[0]$ with $2N + 5$ rows and N_L columns (where $N_L \leq 2N + 5$ is the number of Lyapunov exponents to evaluate): successively multiply by $DM(\mathbf{S}[1]), DM(\mathbf{S}[2]), \ldots, DM(\mathbf{S}[k])$ up to a normalization interval k, perform the explicit QR decomposition, resulting in $\mathbf{Q}[1]$ and $\mathbf{R}[1]$. The LAPACK subroutine library offers standard routines to perform the QR decomposition for nonsquare matrices, needed since typically $N_L \ll 2N + 5$. Accumulate the sum of the logarithm of the N_L diagonal elements of $\mathbf{R}$, and repeat starting with $\mathbf{Q}[1]$. In the asymptotic time limit, the averaged logarithm of $\mathbf{R}$'s diagonal converges to the N_L largest Lyapunov exponents. This requires QR decompositions of a large full-rank matrix, but it is not required frequently. The calculation time is dominated by the evaluation of $DM(\mathbf{S}[k])$. The overall algorithm is a numerically superior version of the Gram-Schmidt orthogonalization used by Farmer [12]. From the exponents λ_i, we choose K such that $\sum_{i=1}^{K} \lambda_i > 0$ and $\sum_{i=1}^{K+1} \lambda_i < 0$, and form the Lyapunov dimension [11] $D_L = K + \sum_{i=1}^{K} \lambda_i / |\lambda_{K+1}|$. The main free parameter governing the dimension is γ: for $\gamma \geq 0.5$, D_L increases nearly linearly with γ. At $\gamma = 1$, the Lyapunov dimension is about 22.

We investigate the synchronization of two $\gamma = 1$ IRLs. The coupling is unidirectional: the unaltered output of a transmitter feeds into a receiver; this is depicted in Fig. 1. After investigating a variety of coupling schemes we settled on the following:

$$\begin{aligned}
\zeta_1(t+1) &= f[\zeta_1(t), w_1(t)], \\
dw_1(t)/dt &= g[w_1(t), |\zeta_1(t)|^2], \\
\zeta_{\mathrm{Comb}}(t) &= (1-c)\zeta_2(t) + c\zeta_1(t), \\
\zeta_2(t+1) &= f[\zeta_{\mathrm{Comb}}(t), w_2(t)], \\
dw_2(t)/dt &= g[w_2(t), |\zeta_{\mathrm{Comb}}(t)|^2].
\end{aligned}$$

The free running transmitter has state variables $\zeta_1(t)$ and $w_1(t)$. Into the receiver we inject the transmitted electric field $\zeta_1(t)$, scaled by c, along with a fraction $(1-c)$ of $\zeta_2(t)$. When $\zeta_1(t) = \zeta_2(t)$ the equations are precisely the same. Ordinary synchronization is thus possible, and generalized synchronization [13] is not required. It is not physically possible to couple the population inversion dynamics, so no information about $w_1(t)$ is transmitted to the receiver. When c was larger than ≈ 0.3, synchronization was observed for a wide range of parameter values.

We computed the mean time to achieve synchronization over an ensemble of initial conditions. Two IRLs were run with $c = 0$ for a sufficiently long time for each to reach asymptotic behavior. The coupling was then turned on. We evaluated the time t_s past which $|\zeta_2(t) - \zeta(t)|$ remained less than some small value for all t up to some large T. If the systems do not synchronize, the average of t_s over the ensemble, $\langle t_s \rangle$, would be T. Synchronization is exhibited by $\langle t_s \rangle \ll T$. Simultaneously, we computed the CLEs of $[\zeta_2(t), w_2(t)]$, evaluated along the synchronization manifold, $\zeta_2(t) = \zeta_1(t), w_2(t) = w_1(t)$, with $\zeta_1(t)$ and $w_1(t)$ treated as given in the evaluation of the Jacobian. The results of our computations are shown in Fig. 2(a).

In Fig. 2(b) we plot the largest CLE and the average time to synchronization as a function of c. A largest CLE below zero is necessary, but not sufficient, to guarantee synchronization [14]. In our system, the difference between them was small: the largest CLE becoming negative is a good predictor of the boundary of synchronization. Though not shown, $w_1(t)$ and $w_2(t)$ also fully synchronized: $|w_1(t) - w_2(t)|$ converged to zero.

With $c = 1$ the receiver is an "open loop" demodulator of any signal modulated onto $\zeta_1(t)$. In this special case, we may prove that $|w_1(t) - w_2(t)|$ approaches zero faster than $e^{-2\gamma t}$, and when this occurs, $\zeta_1(t) \longrightarrow \zeta_2(t)$ too. Write the equations for two coupled IRLs with $c = 1$,

$$\begin{aligned}
\zeta_1(t+1) &= f[\zeta_1(t) + m(t), w_1(t)], \\
dw_1(t)/dt &= g[w_1(t), |\zeta_1(t) + m(t)|^2], \\
\zeta_2(t+1) &= f[\zeta_1(t) + m(t), w_2(t)], \\
dw_2(t)/dt &= g[w_2(t), |\zeta_1(t) + m(t)|^2],
\end{aligned}$$

where $m(t)$ is added to the electric field amplitude of the first laser. The output $\zeta_1(t)$ is nonlinearly dependent on

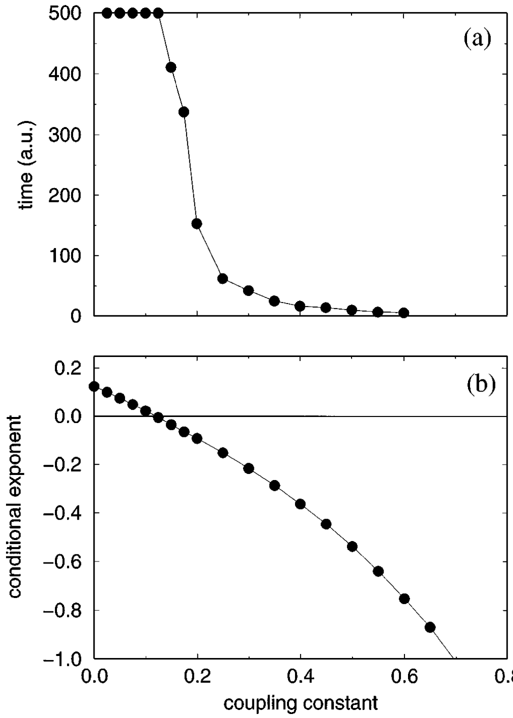

FIG. 2. (a) Mean time to synchronization and (b) largest CLE for $\gamma = 1.0$. The average time to synchronization starts to sharply decrease from the maximum run time (which implies no synchronization) at approximately the same coupling where the largest CLE becomes negative.

$m(t)$. Subtract the differential equations,

$$\frac{d[w_1 - w_2]}{dt} = -2\gamma\{w_1 - w_2 + |\zeta_1 + m|^2 e^{G w_2}(e^{G[w_1 - w_2]} - 1)/G\}.$$

Since $e^A - 1 \geq A$ and $e^A \geq 0$ for real A,

$$\frac{d[w_1 - w_2]}{dt} \leq -2\gamma[w_1 - w_2]\{1 + |\zeta_1 + m|^2 e^{G w_2}\},$$

showing that $|w_1(t) - w_2(t)| \rightarrow 0$ faster than $e^{-2\gamma t}$. We found numerically that the size of the electric field is large enough that even when γ is quite small, as in the erbium case, the product $\gamma|\zeta_1(t) + m(t)|^2$ remains order unity or larger. This means that the communications method suggested by us will be robust against noise contamination which bumps the system off the synchronization manifold. Similar reasoning shows the maps for $\zeta_1(t)$ and $\zeta_2(t)$ also converge: $|\zeta_1(t) - \zeta_2(t)| \rightarrow 0$. This synchronization is independent of the external pumping and the pumping rate Q, so the application of this result to the equations for the erbium ring laser is direct.

The second laser receives $\zeta_1(t) + m(t)$ and produces $\zeta_2(t) = \zeta_1(t)$ in its optical cavity. This allows us to recover $m(t)$ from the difference between ζ_2 and the transmitted signal. The basic idea works for any invertible combination of ζ_1 and $m(t)$. In a real connection between lasers the properties of the optical fiber channel must be considered, but here we have taken it as a perfectly transmitting, lossless, dispersion free medium. We implemented this idea using a sample of speech from one of the authors. In Fig. 3 we show the message (upper panel), the actual transmitted $\mathrm{Re}[\zeta_1(t)]$ (central panel), and the recovered message (lower panel) for both perfect parameter matching (solid line) and a 1% difference in γ between transmitter and receiver (dashed line). Even with parameter mismatch, fidelity is good. We have observed successful recovery of messages using significantly smaller c using the obvious generalization of the modulation method to closed loops, starting with $c \geq 0.3$. Also using binary messages of ± 1 and a suitable digital decision rule, we have found bit error rates $\leq 10^{-5}$ even with mismatches in γ of as much as 50%.

In summary, we have developed a theoretical model for an "Ikeda ring laser" which is an unpolarized representation of the essential dynamics in an erbium-doped fiber ring laser. We have successfully synchronized two such systems described by delay-differential equations, even in the deeply chaotic regime, using an experimentally accessible coupling scheme. This work is a prelude to investigating high bandwidth information transmission utilizing chaos in coupled ring-laser systems. It also confirms that synchronization and information transmission using high-dimensional optical chaos is mathematically feasible. Finally, we note again that VanWiggeren and Roy [6] have experimentally verified the essentials of our methods for synchronization of and communication with chaotic ring lasers.

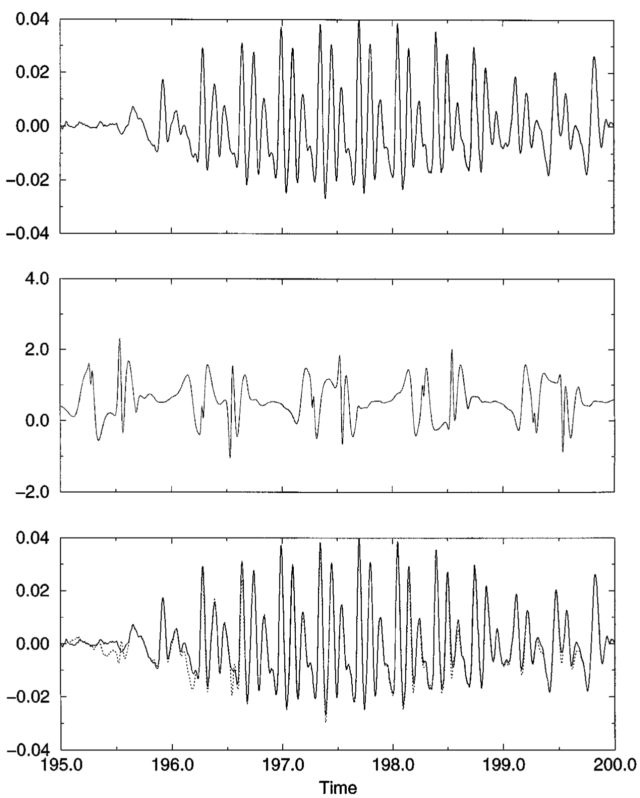

FIG. 3. A segment of speech (upper), $\mathrm{Re}[\zeta(t)]$ as modulated by the message and sent from transmitter to receiver (middle), the recovered message (lower). In the lower panels the solid line is for exact parameter match between transmitter and receiver; the dashed line results when γ in the receiver is 1% larger than γ in the transmitter.

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