

## How Sandcastles Fall

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Capillary forces significantly affect the stability of sandpiles. We analyze the stability of sandpiles with such forces, and find that the critical angle is unchanged in the limit of an infinitely large system; however, this angle is increased for finite-sized systems. The failure occurs in the bulk of the sandpile rather than at the surface. This is related to a standard result in soil mechanics. The increase in the critical angle is determined by the surface roughness of the particles, and exhibits three regimes as a function of the added-fluid volume. Our theory is in qualitative agreement with the recent experimental results of Hornbaker *et al.*, although our interpretation differs. [S0031-9007(98)05798-6]

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The continuum mechanics of most materials was established in the 19th century; however, the mechanics of granular materials is still largely mysterious [1]. The study of granular media is also motivated by the ubiquity of this form of matter in a variety of industrial contexts, as well as in geophysical ones.

While most recent attention has focused on dry granular media, a recent experimental study by Hornbaker *et al.* has opened the relatively unexplored subject of “humid” granular media, in which small amounts of added fluid generate, through capillarity, adhesive forces between the grains [2]. Somewhat whimsically, these authors argue that their work is appropriate for the understanding of sandcastles; we actually take this point seriously, because adhesive forces and other liquid effects are extremely important in geophysical applications, of which sandcastles are an unusual example.

In this Letter, we present a theory of the stability of humid sandpiles, based upon a continuum analysis of their statics. While previous work on the statics and dynamics of dry sandpiles concentrated on the behavior of the pile’s surface [3], we find that the addition of small adhesive forces between the grains causes the site of failure to move from the surface into the bulk of the sandpile, a well-known fact in soil mechanics. Even though the failure of the sandpile at the critical angle occurs at some finite depth, in the limit of infinite-system size the critical angle is actually unchanged by the adhesion. For finite systems the angle of repose is increased from the infinite-system/nonadhesive critical angle. By analyzing the cohesive effect of small amounts of wetting fluid, we find that this increase in the critical angle as a function of the added-fluid volume exhibits a range over which the dependence is linear, in agreement with the principal result of Hornbaker *et al.* However, we disagree with the suggestion of these authors that most of the wetting fluid will be found outside the particle contact zones.

To determine the stability of a sandpile, we must have a criterion for local failure of the sandpile. For a nonadhesive (dry) sandpile, a simple phenomenological

criterion for failure is that

$$\tau > k\sigma, \quad (1)$$

where  $\tau$  is the tangential stress across some plane interior to the sandpile,  $\sigma$  is the normal compressive stress across that plane, and  $k$  is the internal friction coefficient. For a given stress state stability requires that there be no plane for which the ratio of  $\tau$  to  $\sigma$  exceeds  $k$ .

Consider a semi-infinite dry sandpile, whose surface is oriented at an angle  $\theta$  to the horizontal. We choose an  $x - z$  coordinate system, in which  $z$  gives the distance from the surface of the pile ( $z > 0$  down) and  $x$  gives the distance parallel to the surface. Then the stress tensor  $\sigma_{ij}$  satisfies the static equations

$$\partial_z \sigma_{zz} + \partial_x \sigma_{xz} = \rho g \cos \theta, \quad (2)$$

$$\partial_z \sigma_{zx} + \partial_x \sigma_{xx} = \rho g \sin \theta, \quad (3)$$

where  $\rho$  is the density of the sandpile. To solve these equations, we first restrict ourselves to solutions which are functions of  $z$  alone—in a semi-infinite system, any  $x$  dependence of the solutions would be liable to generate arbitrarily large stresses near the surface, which would cause the system to buckle. The most general  $z$ -dependent solution, which also satisfies the boundary condition that the surface is stress free, is

$$\sigma_{zz} = \rho g z \cos \theta, \quad (4)$$

$$\sigma_{xz} = \rho g z \sin \theta, \quad (5)$$

$$\sigma_{xx} = C(z). \quad (6)$$

The well-known stress indeterminacy in granular media implies that  $C(z)$  is at this stage an unknown function. There have been a number of proposed closure schemes for the stress equations, including elastic, rigid-plastic, and elastoplastic constitutive relations [4]. Beyond this variety of methods of closing the stress equations, it is not even clear that our problem is well posed. The stress

state, and thus the critical angle, may well depend on the manner of preparation of the medium.

Nevertheless, there is a classical method of obtaining an upper bound on the critical angle, due to Mohr and Coulomb, which we also expect to provide, at the very least, a good estimate of this angle. This method is to find the form of  $C(z)$  which *maximizes* the critical angle. The angle thereby obtained will be an upper bound on the true critical angle; furthermore, if the true constitutive relations allow the sandpile to adjust the stress  $C(z)$  within some range (by local rearrangement of grains, or hysteretic frictional effects), we expect that this upper bound will be close to the critical angle. We briefly review the Mohr-Coulomb method, and then apply it to the humid case.

Clearly the function  $C(z)$  that will maximize the critical angle will have the form  $C(z) = C'z$ . We first determine the principal stresses, which are the eigenvalues of the stress tensor. From Eqs. (4)–(6) we find that these eigenvalues are

$$\sigma_{1,2} = z\rho g \cos \theta \left( \frac{1+c}{2} \pm \sqrt{\frac{(1+c)^2}{4} + \tan^2 \theta - c} \right), \quad (7)$$

where  $c = C'/\rho g \cos \theta$ . Both of these eigenvalues depend linearly on  $z$ , the depth below the surface (in the direction normal to the surface). The maximum value of  $\tau/\sigma$  as a function of  $c$  and the plane across which this ratio is computed is

$$\left. \frac{\tau}{\sigma} \right|_{\max} = \sqrt{\frac{(1+c)^2}{4(c - \tan^2 \theta)} - 1}. \quad (8)$$

Now this quantity must be  $< k$ , so to find the maximum critical angle, we minimize the right-hand side of Eq. (8) with respect to  $c$ . We then find that at this minimum value of  $c$ ,

$$\left. \frac{\tau}{\sigma} \right|_{\max} = \tan \theta, \quad (9)$$

so that the critical angle is  $\theta_c = \tan^{-1} k$ , which is the classical Mohr-Coulomb result [5].

Now consider a sandpile in which a normal adhesive stress  $s_A$  is exerted across every plane, in addition to whatever other stresses may exist due to the body forces. This stress introduces a normal force between pairs of contiguous particles which allows the sandpile to support a finite shear stress, even in the limit of zero applied compressive stress. The maximum supported shear stress, in this case, is  $ks_A$ , and we therefore replace the dry sand failure criterion, Eq. (1), by

$$\tau > k(\sigma + s_A). \quad (10)$$

Performing a calculation similar to the one above, we find that the failure criterion, which is now an explicit

function of depth, is

$$k = \tan \theta \left( 1 + \frac{s_A}{\rho g z \cos \theta} \right)^{-1}. \quad (11)$$

Note that our Mohr-Coulomb analysis, now local, applies only at the incipient failure depth and does not determine the global stress state of the sand pile.

The criterion thus derived, Eq. (11), is most stringent as  $z \rightarrow \infty$ , in which case the dry sandpile result  $\theta_c = \tan^{-1} k$  is recovered. On the other hand, for a sandpile of fixed depth  $D$ , the failure must occur at most at depth  $D$ . Thus the critical angle will be the solution of the equation

$$k = \tan \theta_c(D) \left( 1 + \frac{s_A}{\rho g D \cos \theta_c(D)} \right)^{-1}, \quad (12)$$

giving a critical angle  $\theta_c(D)$  that decreases monotonically with  $D$ . Thus finite humid sandpiles have a depth-dependent critical angle, unlike dry sandpiles, which is a well-known result in soil mechanics [5]. In addition, we see that humid (i.e., cohesive) sandpiles fail at depth. In the case of small adhesion stress  $s_A/\rho g D \ll 1$ , we can write

$$\tan \theta_c \approx k + \frac{ks_A}{\rho g D} \sec[\tan^{-1}(k)]. \quad (13)$$

If, as we are assuming, the adhesion arises from capillary forces, we must still connect the adhesive stress to the amount of fluid present. We suppose that the sand is composed of macroscopically spherical grains (radius  $R$ ) whose surface roughness may be characterized as follows: the spatial correlation of fluctuations in local surface height saturates at height  $l_R$  at a lateral distance  $d$  that is much smaller than the particle radius,  $d \ll R$ . Since the particles are macroscopically spherical, we require that  $l_R \ll R$  (see Fig. 1).

We can characterize the surface roughness of two particles in contact by considering the function  $\delta(x)$  which gives the average distance between the two particles a lateral distance  $x$  from an asperity at which the two particles are in contact. We write  $\delta(x)$  in the form [6]

$$\delta(x) = l_R f(x/d), \quad (14)$$

where  $f(w)$  is a scaling function with the limits

$$f(w) \sim \begin{cases} w^\chi & w \rightarrow 0, \\ 1 & w \rightarrow \infty. \end{cases} \quad (15)$$

The roughness exponent,  $\chi$ , of the surface satisfies  $0 < \chi \leq 1$ .

Note that this form can only be valid for  $x < \sqrt{l_R R}$ . For larger values of  $x$ , the macroscopic curvature of the particles will determine the local distance between them (see Fig. 1).

For such particles, there are three regimes for the capillary force exerted by a wetting fluid as a function of  $V$ , the total amount of fluid present per particle contact.

*Asperity regime.*—For the smallest values of  $V$ , the capillary force is dominated by the accumulation of fluid around a single or a small number of asperities at which two neighboring particles are in contact. This will hold until the lateral extent of the fluid-filled region exceeds  $d$ , determining a maximum contact fluid volume for this regime,  $V_1 \sim l_R d^2$ .

We can write the adhesion force  $f_A$  as

$$f_A = \frac{\Gamma A}{r}, \quad (16)$$

where  $\Gamma$  is the surface tension of the fluid,  $r$  is the radius of curvature of the meniscus of the fluid layer connecting the two grains near the asperity, and  $A \sim V/\delta$  is the area of the contact patch.  $-\Gamma/r$  is the pressure reduction due to the capillary meniscus. Because  $r$  will be approximately the distance between the particles at the meniscus, we find

$$P \sim -\frac{\Gamma}{l_R} \left( \frac{V}{V_1} \right)^{-\chi/(2+\chi)}, \quad (17)$$

and the adhesion force is

$$f_A \sim \frac{\Gamma V_1}{l_R^2} \left( \frac{V}{V_1} \right)^{(2-\chi)/(2+\chi)}, \quad \text{for } V < V_1, \quad (18)$$

where  $V_1 = l_R d^2$ .

For a rough surface where  $\chi = 1$ , Eq. (18) shows that the force depends on the cube root of the fluid volume. This is identical to the dependence of the cohesive force between a cone and plate on the volume of the liquid bridge connecting them [2]. It is to be expected that the cone-and-plate model will reproduce the cohesive force near a single asperity.

*Roughness regime.*—For larger values of  $V$ , the fluid will occupy a statistically rough region, which is still small enough that the macroscopic curvature of the particles plays no role—however, the fluid occupies more than the volume around a single asperity. This regime occurs for  $V_1 < V < V_2$ , where  $V_2 = l_R^2 R$ . The pressure is

$$P \sim -\frac{\Gamma}{l_R}, \quad (19)$$

and the force will be

$$f_A \sim \frac{\Gamma V}{l_R^2}, \quad \text{for } V_1 < V < V_2. \quad (20)$$

In this roughness regime, the cohesive force is linear in the volume of the added fluid, reproducing the linear dependence found by Hornbaker *et al.*

*Spherical regime.*—When the lateral extent of the fluid contact exceeds  $d$ , then the wetting region will be determined by the macroscopic curvature of the particles, and the surface roughness will no longer play a significant

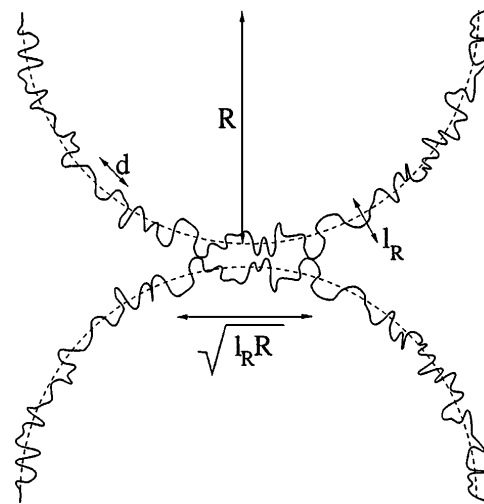


FIG. 1. The contact zone between two rough particles of radius  $R$ . The scale of height deviations from the mean is  $l_R$ , and the height fluctuations are correlated over a distance  $d$ . The lateral size of the contact zone in which the macroscopic curvature of the particles is not apparent is  $\sim \sqrt{l_R R}$ .

role. In this case the pressure is given by [7]

$$P = -\frac{\Gamma}{\sqrt{V/2\pi R}}, \quad (21)$$

and the force by

$$f_A = 2\pi\Gamma R, \quad \text{for } V > V_2, \quad (22)$$

which is *independent* of the volume of the liquid bridge joining the two grains. Thus the linear increase of the cohesive force with fluid volume saturates for volumes  $V > V_2 = l_R^2 R$  (see Fig. 2).

If the fluid wets the surface of the particles, then, in addition to the fluid in the contact region, there will

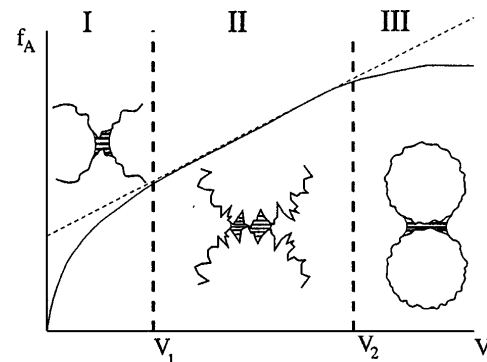


FIG. 2. The behavior of the adhesive force between two rough, “spherical” particles. The three regimes of the force vs volume of the wetting layer are I, asperity regime; II, roughness regime; and III, spherical regime. The insets show the extent of the wetting region typical of each regime.

also be a layer of fluid of thickness  $t$  on the surface of the particles. Typically, this film will be no thicker than a few monolayers; hence, there are a complicated set of forces between this film and the surface. To simplify, we consider only the van der Waals forces, which for a thickness  $t$  generate a “disjoining pressure”  $P_d$  given by [8]

$$P_d = \frac{2H}{t^3}, \quad (23)$$

where  $H$  is the Hamaker constant ( $H < 0$  for a wetting fluid). To determine the thickness of the wetting region, we must set this disjoining pressure equal to the pressure inside the contact regions. If the radius of curvature of the contact meniscus is  $r$ , then

$$t = \left( -\frac{2Hr}{\Gamma} \right)^{1/3}. \quad (24)$$

In the asperity regime,  $t$  will increase with the meniscus radius of curvature. In the roughness regime, however,  $r$  saturates at a value  $\sim l_R$ , and  $t$  will be constant. In this regime any added fluid enters the contact region. Finally, in the spherical regime,  $t$  will again increase.

Now consider a system with a volume  $V_l$  of liquid per particle. If we suppose that the spheres are close packed, then each sphere has 12 neighbors, so there is an average of 6 contacts per sphere, each with a fluid volume of  $V = V_l/6$ . The average number of contacts per unit area will be  $(3\phi_V/\pi R^2)$ , where  $\phi_V$  is the volume fraction of the particles. Thus for a close-packed lattice, for which  $\phi_V = \sqrt{2}\pi/6$ , the adhesive stress will be approximately

$$s_A \approx \frac{f_A}{\sqrt{2}R^2}, \quad (25)$$

and we can now substitute into Eq. (13) to obtain the dependence of the critical angle upon fluid volume. For a sandpile of fixed depth  $D$ , this is linear in  $V$ ,

$$\tan \theta_c \approx k + \frac{k\Gamma V}{\sqrt{2}l_R^2 R^2 \rho g D} \sec[\tan^{-1}(k)], \quad (26)$$

up to a saturation result determined by (see Fig. 2)

$$\tan \theta_c \approx k + \frac{\sqrt{2}\pi k\Gamma}{R\rho g D} \sec[\tan^{-1}(k)]. \quad (27)$$

Now let us consider the experiment of Hornbaker *et al.* They measured the critical angle for a medium composed of radius  $4 \times 10^{-2}$  cm polystyrene spheres with small amounts of added oil using the draining crater method [2]. They found a linear increase in the critical angle measured as a function of the volume of added oil. They

claimed that the failure of their systems was at the surface, and concluded that they could account for their results by assuming that 99.9% of the fluid was outside of the contact zones between the particles. Their particles had a surface roughness on the order of  $1 \mu\text{m}$ .

We find that the increase in the critical angle is linear with the fluid volume in the roughness regime, up to the saturation result Eq. (27). We expect the Hamaker constant for a wetting fluid to be negative, and of the order of magnitude of  $H \sim 10^{-20}$  erg, so no more than a few monolayers of fluid should be present on the particle surfaces. Furthermore, in the linear regime, all fluid added to the system will enter the particle contacts. Thus we disagree with the claim that the overwhelming majority of the fluid in this case will be outside the contact zones. Finally, we disagree with the interpretation of their experiment, according to which surface failure is the most relevant failure mode—the failure plane in cohesive materials should be at depth.

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