New Algebraic Approach to Scattering Problems

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Quantum scattering systems described by Hamiltonians which are constructed from the Casimir operators of certain noncompact groups G are considered. We obtain the following result: If U^{χ} and $U^{\tilde{\chi}}$ are the Weyl-equivalent representations of the symmetry group G of the dynamical system, the corresponding S matrices are constrained to satisfy $\mathrm{SU}^{\chi}(g) = U^{\tilde{\chi}}(g)S$, for all $g \in G$. This relation enables one to derive S. As applications, the S matrices corresponding to the dynamical groups $\mathrm{SO}_0(p,q)$ are derived. [S0031-9007(98)05542-2]

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In recent years, dynamical group theories have attracted much attention, for it has been discovered that some quantum scattering systems can be described by these theories. An important development which inspired much of the current work in this direction was presented in Refs. [1,2]. One can algebraically determine the S matrix of a scattering system whose Hamiltonian is constructed from the Casimir operator of a noncompact group G. This technique, which is called Euclidean connection, essentially uses [3] the theory of group expansions or deformations. The explicit calculation of the S matrix is achieved by writing the infinitesimal operators of representations of the dynamical group G in terms of those of the asymptotic group G^0 which describes the problem in the absence of interactions. However, due to the absence of a general procedure for the descriptions of such connection formulas [4], it is rather difficult to derive the S matrix using the above mentioned method.

In this Letter, we give a new algebraic description of the S matrix for such scattering systems. We show that the S matrix for systems under consideration can be determined from dynamical symmetry principles without the knowledge of the relation between infinitesimal operators of G and those of G^0.

Suppose a Hamiltonian H of quantum scattering system is constructed from Casimir operator C of some noncompact group G

$$H = f(C)\,. \tag{1}$$

For example, the Hamiltonian of the two-dimensional Coulomb problem $H = p^2/2 + \beta/r$ is related to the Casimir invariant $C = E_0^2 + (E_+E_- + E_-E_+)/2$ of the noncompact group SO(2, 1) is [2]

$$H = -\frac{\beta^2}{2(C + \frac{1}{4})}\,. \tag{2}$$

Recall the generators E_0, $E_\pm$ satisfy the commutation relations $[E_0, E_\pm] = \pm E_\pm$, $[E_+, E_-] = 2E_0$, and they are expressed in terms of the angular momentum M and the Runge-Lentz vector A_i, $i = 1, 2$ as $E_0 = M$, $E_\pm = (2H)^{-1/2}(iA_1 \pm A_2)$. The scattering eigenstates form a basis for the principal series of the unitary irreducible representations (UIR's) of SO(2, 1) (for details, see Ref. [2] and references therein).

Our aim is to determine the S matrix which connects any arbitrary incoming Ψ^{in} to an outgoing Ψ^{out} state

$$\Psi^{\mathrm{out}} = S\Psi^{\mathrm{in}}. \tag{3}$$

The state vectors Ψ^{in} and Ψ^{out} are assumed to satisfy the free Schrödinger equation (corresponding to the same value E of the energy) $H_0\Phi_\alpha = E\Phi_\alpha$, with α being a complete set of variables that commute with H_0.

However, we find it expedient to use a definition of the S matrix in terms of exact states rather than free ones:

$$\Psi^- = S\Psi^+, \tag{4}$$

where $\Psi^\pm$ are the eigenstates of the full Hamiltonian H which are labeled by the same quantum numbers as Φ,

$$H\Psi_\alpha^\pm = E\Psi_\alpha^\pm\,. \tag{5}$$

The relation of the states Ψ^+ and Ψ^- to Ψ^{in} and Ψ^{out} are as follows: If $\Psi^+(t)$ and $\Psi^-(t)$ are the wave packets which are centered about the stationary states Ψ^+ and Ψ^-, respectively, we have

$$\lim_{t\to-\infty}\Psi^\pm(t) = \Psi^{\mathrm{in}}(t), \qquad \lim_{t\to+\infty}\Psi^-(t) = \Psi^{\mathrm{out}}(t)\,.$$

Here, $\Psi^{\mathrm{in}}(t)$ and $\Psi^{\mathrm{out}}(t)$ are wave packets constructed from the free states. In other words, the states Ψ^+ and Ψ^- are the solutions of the Lippman-Schwinger equations.

On the other hand, by the assumption [see Eq. (1)], the state vectors Ψ^+ and Ψ^- are the eigenstates of the Casimir operator C of the symmetry group G:

$$C\Psi^\pm = q\Psi^\pm, \tag{6}$$

where $q = f^{-1}(E)$. Thus, the scattering eigenstates $\{\Psi_\alpha^+\}$ and $\{\Psi_\alpha^-\}$ form the bases for the Weyl-equivalent representation of the algebra **g** of the symmetry group G, which we denote by U^{χ} and $U^{\tilde{\chi}}$, respectively. (The representations U^{χ} and $U^{\tilde{\chi}}$ have the same Casimir eigenvalues. Such representations are called Weyl equivalent.) Moreover, it follows from Eq. (4) that the representations U^{χ} and $U^{\tilde{\chi}}$ are related by a similarity transformation $U^{\tilde{\chi}} = SU^{\chi}S^{-1}$. The S matrix for the system under consideration is then subject to the constraint equation

$$SU^{\chi}(b) = U^{\tilde{\chi}}(b)S, \qquad \text{for all } b \in \mathbf{g}, \tag{7}$$

or

$$SU^{\chi}(g) = U^{\tilde{\chi}}(g)S, \qquad \text{for all } g \in G, \tag{8}$$

Here, $U^{\chi}(g)$ and $U^{\tilde{\chi}}(g)$ are the corresponding representations of the group G. Equation (7) [or Eq. (8)] actually is used in deriving the S matrix.

Now we are in a position to outline our general method for obtaining the S matrix for scattering problems under consideration. In order to determine the S matrix, we can proceed in two ways. If the principal series of the algebra $\mathbf{g}$ in the scattering basis is known, we can get the recurrence relations for the S matrix by applying both sides of Eq. (7) to the basis vectors. By solving the recurrence relations, one can find the explicit form of the S matrix as a function of the parameters specifying the representation of $\mathbf{g}$. An alternative way employs Eq. (8). By using the "mathematical" realization of a principal series of G on a Hilbert space of some functions, it is possible to derive, from Eq. (8), the functional relations for the kernel of operator S which allow one to determine it. This global approach, which is complimentary to the infinitesimal treatment, allows one to obtain the integral expression for the S matrix.

To gain a better understanding of our approach, we first illustrate it for scattering models with the $SO_0(2,1)$ symmetry group. To be able to use Eq. (7) in the computation of the S matrix, we have to know an abstract realization of the principal series of $su(1,1) \cong so(2,1)$ algebra.

We recall that the principal series of $su(1,1)$ are characterized by the pair $\chi = (\rho, \epsilon)$, where ϵ is equal to 0 or $\frac{1}{2}$, while $0 \leq \rho < \infty$. The representations specified by the labels $\chi = (\rho, \epsilon)$ and $\tilde{\chi} = (-\rho, \epsilon)$ are Weyl equivalent. We can take the eigenvector $|\chi; m\rangle$ of E_0, with $m = n + \epsilon$, $n = 0, \pm 1, \pm 2, \ldots$ as the scattering basis of the carrier space of the representation. The principal series of $su(1,1)$ is given by [5]

$$E_0^{\chi}|\chi; m\rangle = m|\chi; m\rangle, \tag{9}$$

$$E_{\pm}^{\chi}|\chi; m\rangle = (\tfrac{1}{2} - i\rho \pm m)|\chi; m \pm 1\rangle, \tag{10}$$

with the Casimir invariant $C = -\frac{1}{4} - \rho^2$.

We are now ready to define the S matrix. Using the relations $SE_0^{\chi} = SE_0^{\tilde{\chi}}$ [see Eq. (7)] and (9), one has $E_0^{\tilde{\chi}} S|\chi; m\rangle = mS|\chi; m\rangle$, and we can conclude that $S|\chi; m\rangle = S_m|\tilde{\chi}; m\rangle$. Let us find the numbers S_m. To this end, we apply both sides of equality $SE_+^{\chi} = E_+^{\tilde{\chi}}S$ to the basis $|\chi; m\rangle$. We then obtain the recurrence relation

$$(\tfrac{1}{2} - i\rho + m)S_{m+1} = (\tfrac{1}{2} + i\rho + m)S_m, \tag{11}$$

which implies

$$S_m = c(\rho)\,\frac{\Gamma(\frac{1}{2} + i\rho + m)}{\Gamma(\frac{1}{2} - i\rho + m)}, \tag{12}$$

where $c(\rho)$ is a constant of modulus $= 1$. The energy-dependent parameter ρ is determined by the relation between the Hamiltonian H and the Casimir invariant C. For example, for the Hamiltonian of Eq. (2), $\rho = \beta/k$.

Note that the operator S does not mix states belonging to different one-dimensional subspaces H_m which are spanned by $|\chi; m\rangle$. This observation suggests that there might exist a class of one-dimensional potentials related to $SO(2,1)$ for which the S matrix is given by the number S_m. This, in fact, is exactly what happens in the "potential group" approach to the scattering problems in which the representations of group G describe scattering states with the same energy but different potential strengths. For example, the group $SO(2,1)$ appears as the potential group for the Pöschl-Teller potential, where $H = -(C + \frac{1}{4})$ and the eigenvalue m of E_0 is associated with the potential strength. The corresponding S matrix is a 2×2 diagonal matrix with elements equal to S_m (for details, see Ref. [1,2,6]).

We now wish to show briefly how our general method, when applied to the particularly simple case of scattering problems with $SO_0(p,q)$ dynamical symmetry, reproduces all of the familiar formulas of Refs. [1–3,6,7]. For this purpose, we restrict ourselves to the maximally degenerate principal series [8,9] of $SO_0(p,q)$. We can, without loss of generality, assume $p \geq q$. Moreover, we examine the general case $q > 1$ and, in conclusion, indicate briefly the difference from the general case when $q = 1$.

The maximally degenerate principal series of $SO_0(p,q)$, whose second-order Casimir operator C is identically a multiple of the unit, while all higher-order Casimir operators are zero, is labeled by the pair (ρ, ϵ) and $C = -(p - q)^2/2 - \rho^2$, where $0 \leq \rho < \infty$ and ϵ takes the value 0 or 1. The representations specified by labels $\chi = (\rho, \epsilon)$ and $\tilde{\chi} = (-\rho, \epsilon)$ are equivalent. The relevant basis on the carrier space of the representation is given by the decomposition with respect to the maximal compact subgroups $SO_0(p,q) \supset SO(p) \times SO(q) \supset \cdots \supset SO(2)$, where each irreducible representation of a subgroup occurs, at most, once in the reduction. The scattering basis states are denoted by $|\chi; lmM\rangle$, $l + m = \epsilon (\mathrm{mod}\ 2)$, where the labels l and m specify the symmetric tensor representations of $SO(p)$ and $SO(q)$, respectively, and M denotes the remaining labels specifying the representations of $SO(p - 1) \times SO(q - 1)$, and so on down the chain.

Since the matrix of representation for the maximal compact subgroup $SO(p) \times SO(q)$ in this basis is diagonal, it follows from (7) that $S|\chi; lmM\rangle = S_{lm}|\tilde{\chi}; lmM\rangle$. Thus, the S matrix is determined by the numbers S_{lm}.

Let us define the numbers S_{lm}. For this purpose, it is sufficient to consider the fulfillment of condition (7) for infinitesimal operator $I^{\chi}_{1,n}$ corresponding to a one-parameter subgroup consisting of hyperbolic rotations in the plane $(1, n)$ with $n = p + q$, namely,

$$SI^{\chi}_{1,n} = I^{\tilde{\chi}}_{1,n} S \,. \tag{13}$$

The operator $I^{\chi}_{1,n}$ acts on the canonical basis according to the formula [8,9]

$$\begin{aligned} I^{\chi}_{1,n}|\chi; lmM\rangle = {} & \Big(-\frac{p+q}{2} + 1 + i\rho - l - m\Big)c_1|\chi; l+1, m+1, M\rangle + \Big(-\frac{p-q}{2} - 1 + i\rho - l + m\Big) \\ & \times c_2|\chi; l+1, m-1, M\rangle + \Big(\frac{p-q}{2} + 1 + i\rho + l - m\Big)c_3|\chi; l-1, m+1, M\rangle \\ & + \Big(\frac{p+q}{2} - 3 + i\rho + l + m\Big)c_4|\chi; l-1, m-1, M\rangle , \end{aligned}$$

where c_i, $i = 1, \ldots, 4$, are positive numbers which depend on l and m. We then apply both sides of the equality (13) to the basis vector $|\chi; lmM\rangle$. As a result, we obtain a system of equations:

$$\begin{aligned} \Big(-\frac{p+q}{2} + 1 + i\rho - l - m\Big)S_{l+1,m+1} &= \Big(-\frac{p+q}{2} + 1 - i\rho - l - m\Big)S_{l,m} \,, \\ \Big(-\frac{p-q}{2} - 1 + i\rho - l + m\Big)S_{l+1,m-1} &= \Big(-\frac{p-q}{2} - 1 - i\rho - l + m\Big)S_{l,m} \,, \\ \Big(\frac{p-q}{2} - 1 + i\rho + l - m\Big)S_{l-1,m+1} &= \Big(\frac{p-q}{2} - 1 - i\rho + l - m\Big)S_{l,m} \,, \\ \Big(\frac{p+q}{2} - 3 + i\rho + l + m\Big)S_{l-1,m-1} &= \Big(\frac{p+q}{2} - 3 - i\rho + l + m\Big)S_{l,m} \,, \end{aligned}$$

with $l + m = \epsilon (\mathrm{mod}\ 2)$. It follows then,

$$S_{lm} = \gamma(\rho) \frac{\Gamma((\frac{p+q-2}{2} + i\rho + l + m)/2)\Gamma((\frac{p-q+2}{2} + i\rho + l - m)/2)}{\Gamma((\frac{p+q-2}{2} - i\rho + l + m)/2)\Gamma((\frac{p-q+2}{2} - i\rho + l - m)/2)} \,, \tag{14}$$

where $\gamma(\rho)$ is a constant of modulus $= 1$. The S matrices for $SO_0(p, 1)$ groups can be obtained from (14) by substituting $q = 1$ and $m = 0$.

These results can be used in either direction. If the symmetry properties of the dynamical systems are considered as the fundamental assumptions of the quantum theory, then the results actually provide the analytic structure of the S matrix from symmetry principles which do not explicitly contain the notion of space or time. There exist a number of interesting results in this way which were used to analyze collisions between heavy ions [10] and nuclear reactions [11]. In another direction, if one associates an integrable model with Lie algebra through the Hamiltonian H, then one can establish the corresponding S matrix. Such realizations were applied to a family of solvable potentials [1,2,6,12].

We end this Letter with an illustration of our second procedure. Our task is greatly simplified by the fact that the S matrices for scattering systems under consideration are related to intertwining operators. We mention that the operator S satisfying the Eqs. (7) or (8) is called the intertwining operator between representations U^{χ} and $U^{\tilde{\chi}}$. The explicit expressions of the intertwining operators for semisimple Lie groups in terms of kernels are introduced in Ref. [13] (see also Refs. [8,9,14]) and have been extensively studied in Refs. [15,16] in a different context. These may be useful in the derivation of integral formulas for S matrices of other classes of finite scattering systems. As an example, let us compute the S matrices for the problems related to principal series representations of $SO_0(3, 1) \approx SL(2, C)$. The UIR of $SL(2, C)$, in which we are interested, are labeled by the pair $\chi = (\rho, \lambda)$, where λ is an integer or half integer and ρ is real. (Note that both of the Casimir invariants $C = J^2 - M^2$ and $C' = JM$ are nonzero; $C = \lambda^2 - \rho^2 - 1$, and $C' = \lambda\rho$.) The representations $\chi = (\rho, \lambda)$ and $\tilde{\chi} = (-\rho, -\lambda)$ are equivalent. The corresponding intertwining operators in terms of kernels can be extracted from the general formula given by Kunz and Stein [13]. Since this was already done in [15] (see section 16), we only state the result [for details, see also Chap. III of [14], where this operator is determined directly from the intertwining relation (8)].

Let the principal series of $SL(2, C)$ be realized on the Hilbert space of square integrable functions $\varphi(z)$ of complex variables z. Then, in this realization, the unitary operator S has the form $S\varphi(z) = \int K(z, z')\varphi(z')\, dz'$ with the kernel given by

$$\begin{aligned} K(z, z') = {} & (2\pi)^{-1} i^{2\lambda} (i\rho + |\lambda|)\, |z' - z|^{2\lambda - 2 - 2i\rho} \\ & \times (z' - z)^{-2\lambda}. \end{aligned}$$

For our purpose, however, it is more convenient to realize the principal series of $SL(2, C)$ on the Hilbert space $L^2_\lambda(\mathrm{SU}(2))$ of square integrable functions $f(u)$ on the group SU(2) which obeys the condition $f(\omega u) = \exp(i\lambda\beta/2)f(u)$, where ω is a 2×2 diagonal matrix with elements $\exp(-i\beta/2)$ and $\exp(i\beta/2)$ [17]. The connection between these two realization are $\varphi(z) = |u_{22}|^{2\lambda-2-2i\rho}(u_{22})^{2\lambda}f(u)$, where u_{ik} are the matrix elements of the matrix $u \in \mathrm{SU}(2)$. Therefore, when the carrier space of the representation is $L^2_\lambda(\mathrm{SU}(2))$, the operator S has the form $Sf(u) = \int K(u,u')f(u')\,du'$ with the kernel given by

$$K(u,u') = i^{2\lambda}(i\rho + |\lambda|)\,|(u'u^{-1})_{21}|^{2\lambda-2-2i\rho} \times [(u'u^{-1})_{21}]^{-2\lambda}.$$

Taking into account the fact that the basis states $|\chi; jm\rangle$ in $L^2_\lambda(\mathrm{SU}(2))$ differ from the matrix elements $D^j_{\lambda m}$ of the UIR of SU(2) only by the factor $|\chi; jm\rangle = \sqrt{2j+1}\,D^j_{\lambda m}(u)$, $-j \le m \le j$, $j = |\lambda|, |\lambda|+1, \ldots$, we arrive at the integral formula for the S matrix:

$$\langle j'm'|S|jm\rangle = \sqrt{(2j+1)(2j'+1)} \times \int K(u,u')D^j_{\lambda m}(u')\overline{D^{j'}_{-\lambda m}(u)}\,du\,du',$$

where du is the normalized invariant measure on SU(2). Consequently, we obtain $\langle j'm'|S|jm\rangle = \delta_{mm'}\delta_{jj'}S_j$, where

$$S_j = i^{2|\lambda|}\,\frac{\Gamma(1+i\rho+j)\Gamma(-i\rho+|\lambda|)}{\Gamma(1-i\rho+j)\Gamma(i\rho+|\lambda|)}.$$

[For calculation of the integral see Ref. [18]].

These results may have applications in a number of scattering problems with SO(3,1) symmetry. One application is in the study of the scattering processes with spin degrees of freedom. Such a spin-dependent scattering problem was discussed in Ref. [19], where the S matrix for the simplest choice (i.e., for the spin-$\frac{1}{2}$ case) has been evaluated by obtaining the explicit wave functions and by studying their asymptotic behavior. It is not difficult to see that, for $\lambda = \frac{1}{2}$, the result in Ref. [19] can be reproduced. The results can also be used to investigate the scattering processes in the Kaluza-Klein monopole field (see Ref. [20], and references therein). In this case, the quantity λ defines electric charge. Thus, it has become clear that, besides its mathematical beauty, the theory of the intertwining operators may provide a method to construct S matrices for models associated with Lie groups [21].

Finally, the method developed here can also be used to analyze the scattering problems of infinitely extended systems (or systems with an infinite number of degrees of freedom) related to the infinite dimensional algebras especially for the conformal invariant field theories in two dimensions (or the string models), where the Virasoro algebra plays the role of the symmetry group of the theory. Work in this direction is in progress, and we hope to report on it soon. The important fact is that the intertwining operator for Virasoro algebra in terms of vertex operators is explicitly given by [22].

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[1] A. Frank and K. B. Wolf, Phys. Rev. Lett. **52**, 1737 (1984).
[2] Y. Alhassid, J. Engel, and J. Wu, Phys. Rev. Lett. **53**, 17 (1984).
[3] A. Frank, Y. Alhassid, and F. Iachello, Phys. Rev. A **34**, 677 (1986).
[4] R. Gilmore, *Lie Groups, Lie Algebras and Some of Their Applications* (Wiley, New York, 1974), Chap. 10.
[5] N. Ya. Vilenkin and A. U. Klimyk, *Representation of Lie Groups and Special Functions* (Kluwer, Dordrecht, 1991), Vol. 1.
[6] Y. Alhassid, F. Gürsey, and F. Iachello, Ann. Phys. (N.Y.) **167**, 181 (1986).
[7] J. Wu, F. Iachello, and Y. Alhassid, Ann. Phys. (N.Y.) **173**, 68 (1987).
[8] N. Ya. Vilenkin and A. U. Klimyk, *Representation of Lie Groups and Special Functions* (Kluwer, Dordrecht, 1993), Vol. 2.
[9] V. F. Molchanov, Mat. Sbornik **81**, 358 (1970) [Math. USSR Sbornik **10**, 333 (1970)].
[10] Y. Alhassid, F. Iachello, and B. Shao, Phys. Lett. B **201**, 183 (1988); K. Amos *et al.*, Phys. Rev. Lett. **64**, 625 (1990); Phys. Lett. B **269**, 19 (1991); Phys. Rev. C **47**, 2827 (1993).
[11] Y. Alhassid and F. Iachello, Nucl. Phys. **A501**, 585 (1989); A. Lepine-Szily *et al.*, Phys. Lett. B **243**, 23 (1990).
[12] J. Wu, Y. Alhassid, and F. Gürsey, Ann. Phys. (N.Y.) **196**, 163 (1989); J. Wu and Y. Alhassid, J. Math. Phys. (N.Y.) **31**, 557 (1990).
[13] R. A. Kunze and E. M. Stein, Am. J. Math. **89**, 385 (1967); G. Schiffmann, Bull. Soc. Math. Fr. **99**, 3 (1971).
[14] I. M. Gelfand, M. I. Graev, and N. Ya. Vilenkin, *Integral Geometry and Representation Theory*, Generalized Functions Vol. 5 (Academic, New York, 1966).
[15] A. W. Knapp and E. M. Stein, Ann. Math. **93**, 489 (1971).
[16] A. W. Knapp and E. M. Stein, Invent. Math. **60**, 9 (1980).
[17] M. A. Naimark, *Linear Representations of the Lorentz Group* (Pergamon, London, 1964).
[18] G. A. Kerimov and Yi. A. Verdiyev, Rep. Math. Phys. **13**, 315 (1978).
[19] P. Levay and B. Apagyi, J. Math. Phys. (N.Y.) **36**, 6633 (1995).
[20] L. Gy. Feher and P. A. Horvathy, in *Symmetry in Science III*, edited by B. Gruber and F. Iachello (Plenum Press, New York, 1989), p. 399.
[21] M. A. Olshanetsky and A. M. Perelomov, Phys. Rep. **94**, 313 (1983).
[22] A. Tsuchiya and Y. Kanie, Publ. Res. Inst. Math. Sci., Ser. A **22**, 259 (1986).