Models for Superfluid ^{3}He in Aerogel

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Several recent experiments find evidence of superfluidity of ^{3}He in 98%-porous aerogel. The primary effect of the aerogel is that it scatters the quasiparticles of ^{3}He. We find that many experimental findings are quantitatively understood by a relatively simple model that takes into account strong inhomogeneity of the scattering on a length scale of 100 nm. [S0031-9007(98)05646-4]

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The discovery of unconventional pairing states in high-temperature superconductors has generated a lot of interest in impurity scattering in these materials. In particular, the inhomogeneity of the scattering has been considered recently [1]. However, both the experimental and theoretical studies are difficult because of the complicated structure of these substances. Recently, a new possibility was opened for studying impurity effects on unconventional pairing states: superfluid ^{3}He in very porous aerogel. This system has many advantages. For example, the pure state of superfluid ^{3}He is absolutely pure in experiments, and it is theoretically very well understood. A crucial parameter, the coherence length ξ_0, can easily be varied within a factor of 5 by varying the pressure. The torsional oscillator experiments [2,3] and NMR experiments [4] measure directly such basic quantities as the superfluid density, the pairing amplitude, and the spin susceptibility.

In this Letter we give theoretical explanations for some of the experimental observations on superfluid ^{3}He in aerogel. As a first attempt we study a model, where the aerogel is assumed to be a homogeneous scatterer of the quasiparticles of ^{3}He. This model gives predictions with a correct tendency, but it is insufficient quantitatively. A "slab model" gives a clue that the inhomogeneity of the scattering is crucial for understanding the discrepancy. Based on that we construct a relatively simple model of inhomogeneous scattering that quantitatively explains both the transition temperature and the pairing amplitude, and predicts an inhomogeneity length scale of 100 nm. We also consider the upper limit for anisotropic scattering set by the NMR measurements.

In the experiments the aerogel fills only 2% of the total volume ($V = 0.02$), and its surface-to-volume ratio is $A = 260\,000\ \mathrm{cm}^{-1}$ [5]. Assuming naively that the material consists of a network of one-dimensional strands, we can estimate from these numbers alone the strand diameter $4V/A = 3$ nm. The distance between strands is $\sqrt{4\pi V}/A = 20$ nm. The mean free path for straight line trajectories is estimated as $\ell = 4/A = 150$ nm. This is also the mean free path for quasiparticles of ^{3}He when the aerogel is filled with ^{3}He at millikelvin temperatures.

Quasiclassical theory.—Because the volume fraction of the aerogel strands (including an inert layer of ^{3}He atoms on the strands) is small, we neglect all effects that are linear in the volume fraction. In particular, we assume that the density, the Landau Fermi-liquid parameters, the coupling constant of the pairing interaction, and the dipole-dipole interaction constant are unchanged from the bulk. The changes of these parameters are of the same order of magnitude as the volume fraction because they arise from processes of relatively high energy and short length scale [6]. Much larger effects on superfluidity arise from processes in the immediate vicinity of the Fermi surface. Scattering of quasiparticles from the aerogel strands modifies the superfluid state within the distance ξ_0, and causes an effect that is proportional to the ratio ξ_0/ℓ, which approaches unity in 98%-porous aerogel. Here ξ_0 is the superfluid coherence length. It is defined by $\xi_0 = \hbar v_{\mathrm{F}}/2\pi k_{\mathrm{B}} T_{c0}$, where T_{c0} is the transition temperature in bulk ^{3}He and v_{F} is the Fermi velocity. ξ_0 is a function of pressure varying between 16 nm (melting pressure) and 77 nm (zero pressure).

Because of the s-wave pairing of conventional superconductors, their T_c and pairing amplitude is nearly unaffected by nonmagnetic scattering at $\xi_0/\ell \sim 1$, only the Ginzburg-Landau (GL) coherence length gets shorter [7]. But in a p-wave superfluid such as ^{3}He, the scattering causes destructive interference and leads to complete depression of superfluidity already at $\xi_0/\ell \sim 1$.

All of the models we discuss are quasiclassical. This means that the aerogel is modeled as a collection of incoherent scattering centers at locations $\mathbf{r}_j$. Each center is assumed much smaller than ξ_0 but, similar to aerogel strands, they can be large in comparison to the Fermi wave length $\lambda_{\mathrm{F}} = 2\pi/k_{\mathrm{F}} \approx 0.7$ nm. For each scattering center, a fully quantum-mechanical treatment is allowed, in principle, but we describe it phenomenologically by phase shifts and scattering cross sections. The interference of different scattering centers leads to weak localization corrections, which are small because aerogel has random structure and $\lambda_{\mathrm{F}}/\ell \ll 1$.

The coherence length ξ_0 is the only pressure dependent length scale in scattering models [8]. This implies that the calculated T_c can be compared with experiments using the scaling presented in Fig. 1. The vertical axis is the suppression of the transition temperature relative to the bulk, T_c/T_{c0}. The horizontal axis is ξ_0 divided by a length L. The scale L is a constant that characterizes each aerogel sample. By definition, L equals $\xi_0(p)$ at the pressure p, where $T_c/T_{c0} = 0.7$. In other words, the horizontal scale is chosen so that all data sets coincide at the point $(1.0, 0.7)$. For the three different samples used in the experiments we find $L = 36$ nm [2], $L = 25$ nm [4], and $L = 24$ nm [3]. With this scaling the three data sets seem rather consistent with each other.

In order to compare the amplitude $\Delta(T, \mathbf{r})$ of the order parameter, we study the suppression factor [4]

$$S_{\Delta^2}(t) = \frac{\langle \Delta^2(tT_c, \mathbf{r}) \rangle}{\Delta_0^2(tT_{c0})} . \tag{1}$$

As before, the subscript 0 refers to the bulk, i.e., to the case of pure ^{3}He. The parameter t denotes the temperature relative to the transition temperature. An average over locations $\mathbf{r}$ is indicated by $\langle \cdots \rangle$.

We can construct a suppression factor S_{ρ_s} for the superfluid density ρ_s in complete analogy with (1). However, ρ_s depends strongly on the Fermi-liquid parameter $F_1^s = 3(m_{\mathrm{eff}}/m - 1)$. Because the pressure dependence of F_1^s spoils the scaling with ξ_0 discussed above, it is preferable to use the suppression factor $S_{\tilde{\rho}_s}$ for the bare superfluid density $\tilde{\rho}_s$ defined by $\rho_s = \tilde{\rho}_s/[1 + \frac{1}{3}F_1^s(1 - \tilde{\rho}_s/\rho)]$, where ρ is the density of the liquid.

The experimental suppression factors are plotted against $(T_c/T_{c0})^2$ in Figs. 2(a) and 2(b). The NMR experiment measures S_{Δ^2} because the dipole-dipole interaction constant g_d [9] is unchanged by scattering. $S_{\tilde{\rho}_s}$ can be extracted from torsional oscillator experiments. The t dependencies of S_{Δ^2} and $S_{\tilde{\rho}_s}$ are qualitatively similar, but are more pronounced in the latter.

Homogeneous scattering model (HSM).—This is the simplest scattering model. The principal assumption is that the scattering probability is independent of the location. Additionally we assume that the scattering medium is isotropic, i.e., ℓ is independent of the direction of quasiparticle momentum. These are just the standard assumptions made in studying impurities in superconductors [7]. We also neglect magnetic scattering because it does not seem to be important for the effects we consider. A convenient property of the isotropic HSM is that both the GL theory and the Leggett's theory of NMR [9] have the same form as in pure ^{3}He. Only the parameters of these theories have different values, as will be discussed below.

The Ginzburg-Landau theory is formulated in terms of a free energy functional of the 3×3 matrix order parameter, $A_{\mu i}$, where μ represents the spin components and i represents the orbital components of the pair state. The "bulk" terms are [10,11]

$$\begin{aligned} f_{\mathrm{bulk}} = {} & \alpha A_{\mu i}^* A_{\mu i} + \beta_1 |A_{\mu i} A_{\mu i}|^2 + \beta_2 (A_{\mu i} A_{\mu i}^*)^2 \\ & + \beta_3 A_{\mu i}^* A_{\nu i}^* A_{\nu j} A_{\mu j} + \beta_4 A_{\mu i}^* A_{\nu i} A_{\nu j}^* A_{\mu j} \\ & + \beta_5 A_{\mu i}^* A_{\nu i} A_{\nu j} A_{\mu j}^* . \end{aligned} \tag{2}$$

The transition temperature T_c is determined by the condition $\alpha(T_c) = 0$. Minimizing (2), one finds the order

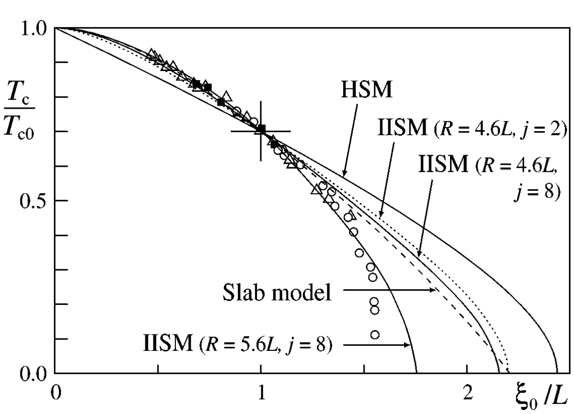

FIG. 1. The transition temperature in aerogel relative to that in bulk, T_c/T_{c0}. The horizontal axis is the coherence length $\xi_0 = \hbar v_F / 2\pi k_B T_{c0}$ divided by L. The scale L is chosen so that the data sets coincide at the cross. The experimental results are from Refs. [2] (△), $L = 36$ nm, [4] (■), $L = 25$ nm, and [3] (∘), $L = 24$ nm. The lines correspond to the homogeneous scattering model (HSM) [12,13], to the slab model [15], and to the isotropic inhomogeneous scattering model (IISM) with different sphere radii R and scattering profile parameters j.

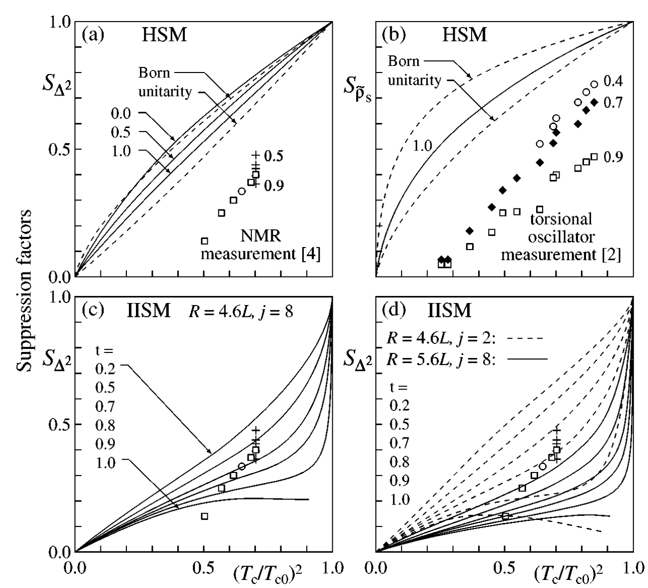

FIG. 2. The suppression factors for the gap (S_{Δ^2}) and superfluid density ($S_{\tilde{\rho}_s}$) as a function of squared T_c suppression, $(T_c/T_{c0})^2$. The upper frames [(a) and (b)] present the experimental data and the results of the homogeneous scattering model. The lower frames [(c) and (d)] are the results of the isotropic inhomogeneous scattering model. The numbers associated with curves and data points denote t [see Eq. (1)]. Born ($\sin^2\delta_0 \to 0$) and unitarity ($\sin^2\delta_0 = 1$) limits are shown by dashed lines in (a) and (b) at $t = 1$. All other curves are for the intermediate case $\sin^2\delta_0 = 0.5$.

parameter amplitudes Δ and free energies f of the various phases. For example, the polar, planar, and B phases have $f = k\alpha\Delta^2/2 = -k\alpha^2/(4k\beta_{12} + 4\beta_{345})$ with $k = 1$, 2, and 3, respectively, where $\beta_{ij...} = \beta_i + \beta_j + \cdots$, and the A phase has $f = \alpha\Delta^2 = -\alpha^2/4\beta_{245}$.

The coefficient α is given by [12]

$$\alpha = \frac{N(0)}{3}\left[\ln\frac{T}{T_{c0}} + \sum_{n=1}^{\infty}\left(\frac{1}{n-\frac{1}{2}} - \frac{1}{n-\frac{1}{2}+x}\right)\right], \tag{3}$$

where $x = \hbar v_F/4\pi T\ell_{\mathrm{tr}}$, ℓ_{tr} is the transport mean free path, and $2N(0)$ is the density of states at the Fermi surface. The suppression of T_c in the HSM is shown in Fig. 1 ($\ell_{\mathrm{tr}} = 8.7L$). Its dependence on ξ_0 is the same as found for magnetic impurities in s-wave superconductors [13].

For the coefficients β_i we make the additional assumption that only s-wave scattering is important, and obtain

$$\begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\\ \beta_4\\ \beta_5\end{pmatrix} = a\begin{pmatrix}-1/2\\ 1\\ 1\\ 1\\ -1\end{pmatrix} + b\begin{pmatrix}0\\ 1\\ 0\\ 1\\ -1\end{pmatrix} + \begin{pmatrix}\Delta\beta_1^{\mathrm{sc}}\\ \Delta\beta_2^{\mathrm{sc}}\\ \Delta\beta_3^{\mathrm{sc}}\\ \Delta\beta_4^{\mathrm{sc}}\\ \Delta\beta_5^{\mathrm{sc}}\end{pmatrix}, \tag{4}$$

$$a = \frac{N(0)}{120(\pi T)^2}\sum_{n=1}^{\infty}(n - \tfrac{1}{2} + x)^{-3},$$

$$b = \frac{N(0)v_{\mathrm{F}}}{288(\pi T)^3\ell}\left(\sin^2\delta_0 - \tfrac{1}{2}\right)\sum_{n=1}^{\infty}(n - \tfrac{1}{2} + x)^{-4}.$$

Besides ℓ, b also depends on the s-wave scattering phase shift δ_0. The effect of this fully quantum-mechanical degree of freedom on the suppression factors is shown by dashed lines in Figs. 2(a) and 2(b). However, calculations taking into account higher partial waves show that this dependence is essentially averaged out for large scatterers [14]. The end result is approximately the same as if the phase shifts were random: $\sin^2\delta_0 \to 0.5$. Therefore we chose $\sin^2\delta_0 = 0.5$ in all other results of the HSM and the isotropic inhomogeneous scattering model.

The suppression factors of the HSM are essentially the same for different superfluid phases. The difference between A and B phases is negligible also in $S_{\tilde{\rho}_s}$, when the average of the tensor $\tilde{\rho}_s$ is used for the A phase. The stability of A and B phases depends on strong coupling corrections $\Delta\beta_i^{\mathrm{sc}}$, which are not known. Assuming they remain constants, the B phase is favored by increasing scattering. No new phases are stabilized.

We conclude the HSM by noting that it works in the right direction for all T_c, Δ, and ρ_s, but quantitatively, on the level of accuracy we are accustomed to in superfluid ^{3}He, it is clearly inadequate.

Slab model.—This model considers ^{3}He in a gap of thickness D between two diffusely scattering planes. The dashed line in Fig. 1 shows T_c calculated in Ref. [15] ($D = 2.95L$). The agreement with measurement is much better than for the HSM. In particular, T_c suppression is quadratic at small ξ_0 compared to being linear in the HSM. Generally, this feature arises from regions that have no scattering nearby, such as the center of the slab. The suppression of $\langle\Delta^2\rangle$ is also in better agreement with experiments than in the HSM [16]. The principal deficiency of the slab model is its strong anisotropy. We estimate (see below) that, in order to be in agreement with the measured NMR shift, the normal direction of the slab has to vary randomly on a length scale that is smaller than the thickness D. This contradiction prompts us to look for a better model.

Isotropic inhomogeneous scattering model (IISM).—Experimentally, the suppression of T_c seems quadratic at small ξ_0 (Fig. 1). This implies that real aerogel has voids, i.e., regions of negligible scattering. In order to construct a model that is feasible in calculations, we make two basic simplifications. (i) Instead of a random distribution of voids, we consider a periodic lattice of them. (ii) The unit cell of this lattice is approximated by a sphere. In more detail, the boundary condition is that a quasiparticle escaping from the sphere will be returned there at the diametrically opposite point but its momentum is unchanged. (A phase shift similar to Bloch wave functions should be added in the case of nonconstant phase.) In addition to the radius R of the sphere, we need to specify how the density $n(\mathbf{r})$ of scattering centers is distributed in the sphere. When $n(\mathbf{r})$ depends only on the radial coordinate, i.e., $n(\mathbf{r}) = n(r)$, the model is completely isotropic. We study polynomial forms $n(r) = c[(r/R)^j - j(r/R)^{j+2}/(j+2)]$. We show results in Figs. 1 and 2(c) and 2(d) for a steep ($j = 8$) and a slow ($j = 2$) profile. The suppression factors are for an inhomogeneously distorted B phase, but extrapolating the experience from two previous models, the A-type phase would be quite similar.

In spite of the inhomogeneity, there is a single well-defined T_c at which the order parameter becomes nonzero. Because of the proximity effect, T_c is determined collectively by the whole sample, although the weight of high-scattering regions far from low-scattering regions is exponentially small. In any event, the transition can be described as "broadened" if $\langle\Delta^2\rangle(T)$ is strongly nonlinear below T_c. This is the case for a slow profile [dashed lines in Fig. 2(d)] and for large T_c/T_{c0}: In Figs. 2(c) and 2(d) this shows up as a strong t dependence of S_{Δ^2}. In contrast, $\langle\Delta^2\rangle(T)$ is nearly linear over the whole temperature range for small T_c/T_{c0}. [In this case, the t dependence of S_{Δ^2} arises mostly from the nonlinearity of the reference $\Delta_0^2(T)$.]

The IISM reduces to the HSM in the limit of small R. This means that the true distribution of the scattering centers is irrelevant as long as the average scattering over a length scale ξ_0 remains the same [1].

We see that the IISM is in much better agreement with experiments than the HSM when $R \approx 5L$ and $j \approx 8$. The magnitude and t dependence of S_{Δ^2} and most of $T_c(\xi_0)$ are well accounted for. There is a small systematic deviation that experimentally both T_c and S_{Δ^2} drop more rapidly with increasing ξ_0 than in the model. We believe that this difference arises from the periodicity assumption in the IISM. In real aerogel there are fluctuations of all length scales, and with increasing ξ_0 the length scale of most relevant fluctuations also increases. This is consistent with the observed deviations which imply an increasing effective R for increasing ξ_0.

The $\rho_s(T)$ measurement [Fig. 2(b)] shows considerably more nonlinearity than the NMR measurement [Fig. 2(a)]. A possible explanation for this is that the structure of the aerogel is different in the two experiments, the former corresponding to a smaller j. In order to confirm such a hypothesis, both samples should be studied with the same measuring technique.

The large scattering fluctuations predicted by the IISM seem to us to be a natural and essentially unique explanation of the measured suppression factors. However, the predicted effective void radius $\approx 0.8R \sim 100\dots150$ nm is very large compared to the estimates based on small angle x-ray scattering [2]. This problem remains open.

Anisotropic HSM.—According to our introductory estimate, the aerogel consists of randomly oriented strands of diameter 3 nm and length $L_a \approx 20$ nm. This type of anisotropy can have several consequences on the superfluid state. It is known that strongly anisotropic scattering, such as in the slab model, can stabilize the A phase, and one can ask if the aerogel strands could do the same. We have studied this in the limit $L_a \lesssim \xi_0$, where we recover the GL functional (2) of the isotropic HSM since the anisotropy is averaged out on the scale ξ_0. However, the coefficients β_i are modified. We find that anisotropic backscattering, preferentially perpendicular to the strands, can stabilize the A-phase at low pressures, where the B phase is otherwise stable [17].

The anisotropy couples to the orbital part of the order parameter. For example, the α term in the GL functional (2) is modified to $\alpha_{ik}A^*_{\mu i}A_{\mu k}$. Let us study this in the A phase, where the orbital and spin parts are described by $\hat{\mathbf{l}}$ and $\hat{\mathbf{d}}$ vectors, respectively. There are two possibilities [18]. (i) For weak anisotropy the dipole-dipole coupling between $\hat{\mathbf{l}}$ and $\hat{\mathbf{d}}$ keeps them aligned to each other. (ii) For strong anisotropy, $\hat{\mathbf{l}}$ is driven to vary randomly on a scale L_0 where $\hat{\mathbf{d}}$ is still nearly constant. Using similar estimates as Imry and Ma [19], we find that the NMR frequency shift in the former state is unchanged relative to HSM, but it is reduced to essentially zero in the latter. Experiments clearly point to the former state [4]. Our scattering estimate also favors this state, but the margin is rather small: If $L_a \approx 50$ instead of 20 nm, the latter state would be favored. The proximity of the transition gives a natural explanation to the observed sudden extinction of the NMR shift as a function of the tipping angle [4,18].

In conclusion, superfluid ^{3}He in aerogel is, in many respects, an ideal system to study impurity effects in unconventional superfluidity. We find, in particular, that the standard impurity model is robust in the sense that large fluctuations in the scattering are needed in order to get such substantial deviations as seen experimentally.

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