Two Scaling Regimes for Rotating Rayleigh-Bénard Convection

V. M. Canuto and M. S. Dubovikov
NASA, Goddard Institute for Space Studies, 2880 Broadway, New York, New York 10025
(Received 24 April 1997)

Using a turbulence model, we derive two scaling regimes for rapidly rotating Rayleigh-Bénard turbulent convection. For $\mathrm{Ra}_* < \mathrm{Ra} < \mathrm{Ra}_{**}$, where Ra_* and Ra_{**} are functions of Ω, the Nusselt number Nu is a function only of the scaling variable $\mathrm{Ra}/\mathrm{Ra}_*$; this corresponds to the first regime. For $\mathrm{Ra} > \mathrm{Ra}_{**}$, Nu is almost unaffected by rotation and satisfies the nonrotating scaling law $\mathrm{Nu} \sim \mathrm{Ra}^\gamma$, $\gamma \sim 1/3$. The two scaling laws are confirmed by existing data. [S0031-9007(97)05276-9]

PACS numbers: 47.27.Te

Turbulent convection under rapid rotation is not only of scientific interest but also of importance to geophysics and astrophysics. The problem is difficult for both theorists and experimentalists, and thus a limited number of experiments [1–3] and numerical simulations [4–6] are available. As for the theoretical aspects of the problem, even nonrotating convection is known to be a difficult problem, as is rotating turbulence alone, as *prima facie* paradoxes have indicated [7]. The main feature of rotating turbulence is the effect of rotation on the energy transfer. The solution of the latter problem [8–10] allowed the paradoxes just mentioned to be solved and the scaling laws governing the case of free decay to be derived. As for nonrotating convection, the main features exhibited by experiments and numerical simulations have recently been derived from a turbulence model [11–14]. The same model [8–9,11–14] is applied here to study the case of rotating Rayleigh-Bénard convection. Specifically, we study the Nusselt number dependence

$$\mathrm{Nu} = \mathrm{Nu}(\mathrm{Ra}, \mathrm{Ta}, \mathrm{Pr}, A), \tag{1a}$$

where $\mathrm{Nu} = F_T L(\Delta T \chi)^{-1}$, $\mathrm{Ra} = g\alpha \Delta T L^3 (\nu\chi)^{-1}$, $\mathrm{Ta} = 4\Omega^2 L^4 \nu^{-2}$, $\mathrm{Pr} = \nu\chi^{-1}$: F_T is the total heat flux, L is the extent of the convective region, ΔT is the temperature difference between the two plates, ν and χ are the molecular viscosity and conductivity, α is the thermal expansion coefficient, and A is the aspect ratio. As in the $\Omega = 0$ case, the full dynamical equations of the turbulence model can be solved numerically but here we present the analytic solutions corresponding to the limiting case of $\mathrm{Ra}, \mathrm{Ta}, A \to \infty$, for both $\mathrm{Pr} \to 0$ and $\mathrm{Pr} \to \infty$. The main features of the solutions can be summarized as follows:

(1) Convection occurs for $\mathrm{Ra} > \mathrm{Ra}_*$; in the case of rigid plates and $A \to \infty$, Ra_* coincides with the critical $\mathrm{Ra_c}$ first derived in [15] from stability analysis, namely,

$$\mathrm{Ra_c} = 8.7\mathrm{Ta}^{2/3}\,(\mathrm{Pr} > 1),$$
$$\mathrm{Ra_c} = 17.4\mathrm{Ta}^{2/3}\,\mathrm{Pr}^{4/3}\,(\mathrm{Pr} < 1). \tag{1b}$$

In the case of a finite aspect ratio A, as it was shown in the Rossby experiment [1], $\mathrm{Ra}_* < \mathrm{Ra_c}$, a finding that was confirmed by more recent data [2] and that was explained in [16,17] as due to surface waves propagating along the side walls of the cell.

(2) Beginning at $\mathrm{Ra} = \mathrm{Ra}_*$, the function (1a) increases sharply with increasing Ra at fixed Ta so that the slope is much larger than $\mathrm{Nu} \sim \mathrm{Ra}^{1/3}$ corresponding to the no-rotation case. The prominent feature is a scaling law which can be formulated as follows:

$$\mathrm{Nu}(\mathrm{Ra}, \mathrm{Ta}, \mathrm{Pr}, A) = \mathrm{Nu}(x, \mathrm{Pr}, A), \tag{2a}$$

where

$$x = \frac{\mathrm{Ra}}{\mathrm{Ra}_*}, \qquad \mathrm{Ra}_* = f(\mathrm{Ta}, \mathrm{Pr}, A). \tag{2b}$$

We show that as $A \to \infty$ and for $\mathrm{Pr} \gg 1$, we predict

$$\mathrm{Nu} = \frac{1}{256} x^3 (1 + 3x^{-1})^4. \tag{3a}$$

In the other regime of $\mathrm{Pr} < 1$, we derive

$$\mathrm{Nu} = x^3, \tag{3b}$$

provided

$$\mathrm{Ra}_* < 3 \times 10^4 \mathrm{Pr}^{-1}. \tag{3c}$$

In other regimes, the solution can be found only with a numerical treatment of the basic equations.

(3) The region of sharp increase of Nu vs Ra, which we refer to as the first scaling regime, stretches until Ra_{**}, a transitional value of Ra which for $\mathrm{Pr} > 1$ and $\mathrm{Pr} < 1$ is given by

$$\mathrm{Ra}_{**} \approx 3\mathrm{Ta}^{3/4}, \qquad \mathrm{Ra}_{**} \approx \mathrm{C}\mathrm{Pr}^{1/2}\mathrm{Ta}^{3/4}, \tag{4}$$

with $\mathrm{C} = (\ln \mathrm{Pr}^{1/2}\mathrm{Ta}^{1/4})^3$. The sharp increase of Ra vs Ra eventually decreases, and Nu joins the nonrotating Nu vs Ra curve, $\mathrm{Nu} \sim \mathrm{Ra}^\gamma$, $\gamma \approx 1/3$; this is the second scaling regime in which Nu is almost independent of Ta. These basic qualitative features are confirmed by experimental data [1,2].

The derivation of the above results is as follows. We begin by considering the basic stochastic equations for the fluctuating velocity and temperature fields u_i, θ in the Langevin-like form [13,14]

$$\frac{\partial}{\partial t} u_i(\mathbf{k}, t) = f_i^t(\mathbf{k}, t) - k^2 \nu_d(k) u_i(\mathbf{k}, t) + f_i^e(\mathbf{k}, t), \tag{5a}$$

$$\frac{\partial}{\partial t} \theta(\mathbf{k}, t) = f_\theta^t(\mathbf{k}, t) - k^2 \chi_d(k) \theta(\mathbf{k}, t) + f_\theta^e(\mathbf{k}, t), \tag{5b}$$

where the first two terms represent the nonlinear interactions in the NSE (Navier-Stokes equations). As discussed in [13,14], different turbulence models beginning with direct-interaction approximation [18] were also represented in the Langevin form, but they differ in the way they represent the dynamic terms ν_d, χ_d as well as the turbulent forcing f^t's. Here, we follow the turbulence model discussed in [13,14]. Numerous predictions of the model were tested against laboratory, direct numerical simulation (DNS) and large eddy simulation (LES), data [11,12,19]. The model has no free parameters. In the present case we do not solve (5a) and (5b) but the resulting equations for the second-order moments which are derived by multiplying (5a) and (5b) by $u_j(\mathbf{k}')$ and $\theta(\mathbf{k}')$. The work done by the turbulent forces f^t is computed to be [13,14]

$$\langle f_i^t(\mathbf{k}) u_j(\mathbf{k}') \rangle = -(8\pi k^2)^{-1} P_{ij} r(k) \frac{\partial E(k)}{\partial k} \delta(\mathbf{k} + \mathbf{k}'), \tag{5c}$$

$$\langle f^\theta(\mathbf{k}) \theta(\mathbf{k}') \rangle = -(4\pi k^2)^{-1} r_\theta(k) \frac{\partial}{\partial k} E_\theta(k) \delta(\mathbf{k} + \mathbf{k}'), \tag{5d}$$

where P_{ij} is the standard projection operator, $E(k)$ and $E_\theta(\mathrm{k})$ are the velocity and temperature variance spectra, and $r(k)$ and $r_\theta(k)$ are the "rapidity" of the energy and variance flows from large to small eddies. The first is given by [13,14]

$$r(k) \equiv \frac{\Pi(k)}{E(k)} = 2 \int_0^k p^2 \nu_t(p)\, dp. \tag{6a}$$

Here, $\Pi(k) = -\partial T(k)/\partial k$ is the energy flux in k space and $T(k)$ is the energy transfer. The formula for $r_\theta(k)$ is analogous with $\nu_t \to \chi_t$. The latter are the turbulent components of the dynamical viscosity ν_d and χ_d which, in the $\Omega = 0$ case, are given by [13,14]

$$\nu_d(k) \equiv \nu_t(k) + \nu = (\nu^2 + \frac{2}{5} \int_k^\infty p^{-2} E(p)\, dp)^{1/2}. \tag{6b}$$

Rotation hinders energy transfer as indicated by the reduction in $r(k)$ [8,9]

$$r(k) \longrightarrow r_\Omega(k) = r(k)[1 + \Omega^2 (k^2 \nu_d)^{-2}]^{-1/2}, \tag{7a}$$

and $r_\Omega(k)$ in turn changes $\nu_d(k)$ to $\nu_d^\Omega(k)$,

$$\nu_d(k) \longrightarrow \nu_d^\Omega(k) = \nu + \nu_t^\Omega(k) = \nu + \frac{1}{2} k^{-2} \frac{\partial}{\partial k} r_\Omega(k). \tag{7b}$$

The transformations (7) are discussed in detail in [8,9] where solutions of the resulting equations are checked against available data. The function $\chi_d^\Omega(k)$ is obtained through the differential equation ($\tau \equiv \chi_d^\Omega$, $\xi \equiv \nu_d^\Omega$)

$$\frac{d\tau}{d\xi} = \frac{10}{3} \xi (\xi + \tau)^{-1}, \tag{8}$$

with the initial condition $\chi_d^\Omega(\nu) = \chi$. The external forcing in (5a) and (5b) are derived from the original NSE. For a buoyancy driven flow under rotation, we have [11,12]

$$f_i^e(\mathbf{k}) = -\alpha g_j P_{ij}(\mathbf{k}) \theta(\mathbf{k}) - 2[\Omega_\mathbf{k} \times \mathbf{u}(\mathbf{k})]_i, \tag{9a}$$

$$f_\theta^e(k) = \beta_i u_i(\mathbf{k}). \tag{9b}$$

The first term in (9a) represents buoyancy, $\Omega_\mathbf{k} = k^{-2}(\mathbf{k} \cdot \mathbf{\Omega})\mathbf{k}$ and $\beta_i = -\partial T/\partial x_i$.

The general procedure to set up the dynamic equations is as follows [11,12]. One begins by considering an homogeneous flow in which β_i is constant and all second-order moments are proportional to $\delta(\mathbf{k} + \mathbf{k}')$, as in Eqs. (5c) and (5d). Next, one multiplies Eqs. (5a) and (5b) by $u_j(\mathbf{k}')$ and $\theta(\mathbf{k}')$. This leads to a closed system of equations governing the time dependent evolution of the k-space densities of kinetic energy, heat flux, etc. [if one further multiplies the latter by k^2 and integrates over the directions of $\mathbf{k}$, one obtains the spectra $E(k)$, $E_\theta(k)$, etc.]. The equations are linear in the densities but with nonlinear coefficients, e.g., $\nu_d^\Omega(k)$. To extend the system of equations to the case of inhomogeneous convection, one must consider that the spectral densities are now a function of z (distance to the nearest plate), include diffusion terms, and consider that k_z can take only discrete values for rigid plates, $k_z = \pi n/2z$. The diffusion terms make the analytic study of the basic equations rather difficult. On the other hand, numerical solutions for the $\Omega = 0$ case have shown [11,12] that the effect of diffusion is not larger than about 30% and that it decreases with increasing Ra, and thus it does not influence the asymptotic Nu vs Ra relation. On that basis, we shall assume that the same holds true when $\Omega \neq 0$. We shall then study the behavior of the equations without diffusion. As numerical solutions [12] show, in the $\Omega = 0$ case the spectra $E(k)$ and $E_\theta(k)$ have a maximum at the same value of k, say, k_*. We assume that the same holds true when $\Omega \neq 0$. Indeed, it is natural to expect that the maxima of $E(k)$ and $E_\theta(k)$ are close to the maximum of the spectrum of the heat flux $J(k)$. This makes the work of the turbulent forces, Eqs. (5c) and (5d), vanish at k_*. In the stationary case, this makes the equations for the spectral densities homogeneous. A solution exists if the determinant vanishes, which results in either of the two relations

$$\tilde{\nu}^2(k_*)\tilde{\chi}(k_*) - \tilde{\nu}(k_*)g\alpha\beta\sin^2\eta + 4\Omega^2\tilde{\chi}(k_*)\cos^2\eta = 0 \tag{10a}$$

or

$$2\tilde{\nu}(k_*)[\tilde{\nu}(k_*)+\tilde{\chi}(k_*)]^2 - g\alpha\beta\,[\tilde{\nu}(k_*) + \tilde{\chi}(k_*)] \times \sin^2\eta + 8\Omega^2\tilde{\nu}(k_*)\cos^2\eta = 0\,, \tag{10b}$$

where $\tilde{\nu}(k) = k^2\nu_d^\Omega(k)$ and analogously for χ; η is the angle between the z axis and $\mathbf{k}$. We consider in detail the $\mathrm{Pr} \gg 1$ case (for example, Pr = 6.6 in water). The main temperature gradient occurs near the plates $0 < z < z_b$ where turbulence vanishes. Consider a region slightly above z_b where turbulence is still rather weak in the sense that ν_d^Ω and χ_d^Ω are close to ν and χ which we use in Eq. (10a) to obtain

$$\beta = (g\alpha\mathrm{Pr})^{-1}\sin^{-2}\eta(\nu^2k_z^4\cos^{-4}\eta + 4\Omega^2\cos^2\eta)\,. \tag{11}$$

This result yields a multiplicity of solutions for β at fixed $z, g, \alpha, \mathrm{Pr}, \Omega$ and for different η and k_z. In practice, only one of them is stable and coincides with the stationary solution of the time dependent equations for the second-order moments obtained from the basic Eqs. (5a) and (5b). As we have shown for the $\Omega = 0$ case [12], the stable solution of (10a) corresponds to the maximum of the convective flux. We assume that the same holds true for the $\Omega \neq 0$ case. In some form, this requirement is related to the Malkus hypothesis [20]. We choose η and k_z to maximize the convective flux. This occurs when the conductive flux $\chi\beta$ is minimum, which in turn corresponds to a minimum β. Minimizing (11) with respect to η and k_z leads, for large Ω's, to $k_z = k_{z_0}$, where $k_{z_0} = \pi(2z)^{-1}$ is the lowest value of k_z,

$$\cos^2\eta = 2^{-1/3}\left(\frac{\nu}{\Omega}\right)^{2/3}k_{z_0}^{4/3}\,, \tag{12}$$

and thus

$$\frac{1}{3}\,\beta = 4^{1/3}(g\alpha\mathrm{Pr})^{-1}\nu^{2/3}(k_{z_0}\Omega)^{4/3}. \tag{13}$$

Condition (10b) would yield a much larger β. At the boundary, $z = z_b$, the function (13) equals the temperature gradient in the boundary layer, T_b/z_b. This leads to

$$z_b = \frac{27}{4}\,\pi^4(g\alpha\mathrm{Pr}T_b)^{-3}\nu^2\Omega^4\,, \tag{14}$$

and thus

$$F_T = \chi\frac{T_b}{z_b} = \left(\frac{27}{4}\,\pi^4\right)^{-1}(g\alpha)^3(\mathrm{Pr}^{-1}T_b)^4\Omega^{-4}. \tag{15}$$

The Nusselt number is then given by

$$\mathrm{Nu} = \frac{64}{27}\,\pi^{-4}\left(\frac{T_b}{\Delta T}\right)^4\mathrm{Ra}^3\mathrm{Ta}^{-2}. \tag{16}$$

If we represent $\Delta T = 2T_b + \Delta T_c$, to compute ΔT_c we notice that for $\mathrm{Pr} > 1$ the main contribution occurs in the regions where turbulence is still weak in the sense that ν_d^Ω and χ_d^Ω are close to ν and χ so that β is given by Eq. (13) which can be rewritten, using (14), as follows:

$$\beta(z) = T_b z_b^{1/3}z^{-4/3}, \tag{17}$$

which in turn leads to

$$\frac{1}{2}\Delta T_c = \int_{z_b}^{1/2L}\beta(z)\,dz \approx 3T_b[1 - (2z_bL^{-1})^{1/3}]. \tag{18}$$

Equation (18) lacks a contribution δT_c from the region where χ_d^Ω is much larger than χ. It can be obtained by first obtaining $\beta(z)$ from Eq. (10a) with [8,9]

$$\nu_d^\Omega = \sigma_t\chi_d^\Omega = \left(\frac{8}{45}\right)^{1/2}\left(\frac{\epsilon}{\Omega}\right)^{1/2}k^{-1} \tag{19}$$

and then integrating $\beta(z)$. The result is

$$\delta T_c = 0.013\;\Delta T\,\mathrm{Ra}\;\mathrm{Ta}^{-3/4}\ln(L/\ell_d)\,, \tag{20}$$

where $\ell_d = (\nu^3\epsilon^{-1})^{1/4}$ is the dissipation length scale. As we discuss below, $\delta T_c < \Delta T_c$. Substituting Eq. (18) into (16), we obtain

$$10^6\mathrm{Nu} = 5.9y^{-3}(1 + 26.1y)^4, \qquad y \equiv \mathrm{Ra}^{-1}\mathrm{Ta}^{2/3}. \tag{21}$$

Because $\mathrm{Nu} \geq 1$, we obtain from the last formulas the condition

$$\mathrm{Ra} > \mathrm{Ra}_*, \qquad \mathrm{Ra}_* = 8.7\mathrm{Ta}^{2/3}, \tag{22}$$

where Ra_* is the critical Ra for which (21) yields Nu = 1. Not surprisingly, Ra_* coincides with the result Ra_c of the linear stability analysis for an infinite aspect ratio [15]. In terms of Ra_* Eq. (21) can be represented in the form (3a) and (2b). We recall that this result is obtained for strong rotation when the second term in the parentheses in (11) is larger than the first one at $\cos\eta \sim 1$. In the opposite case, rotation produces only a small correction to the asymptotic formula for the $\Omega = 0$ case,

$$\mathrm{Nu} = 0.078\mathrm{Ra}^{1/3}, \tag{23}$$

which is valid for large Ra, A, and Pr. Using the results of Sec. VI of [12], one can derive the condition for which Nu is close to the value (23). We have

$$\mathrm{Ta} < 0.23\mathrm{Ra}^{4/3}. \tag{24}$$

This result yields the transitional value of Ra which divides the Ra axis into two regions characterized by different scaling laws for the function Nu(Ra, Ta). In the interval $\mathrm{Ra}_* < \mathrm{Ra} < \mathrm{Ra}_{**}$, the scaling law (2) holds true. For $\mathrm{Ra} > \mathrm{Ra}_{**}$, Nu almost does not depend on Ta and is close to (23). These qualitative features are confirmed by existing laboratory data [1,2] for water $\mathrm{Pr} = 6.6$. The Nu vs Ra data (Fig. 21 of Ref. [2]) show the existence of two regimes: for $\mathrm{Ra} > \mathrm{Ra}_{**}$, Nu is close to the $\Omega = 0$ result, while for $\mathrm{Ra} < \mathrm{Ra}_{**}$ the curves for different values of Ta parallel one another. This implies that if plotted against the ratio $\mathrm{Ra}/\mathrm{Ra}_*$, they would collapse into a single curve.

As for the $\mathrm{Pr} \ll 1$ case, Eqs. (3b) and (3c) and the second of (4) were derived in an analogous manner, the main differences being that the minimum value of β is obtained from (10b) rather than from (10a) and that the major contribution to ΔT is $\approx 2T_b$ which arises from the near wall region where conduction dominates and $\chi > \chi_t$. At the same time, in most of the region $\nu_t \gg \nu$ so that (3b) depends critically on the form of transfer which is expressed through Eqs. (6) and (7). In the $\mathrm{Pr} \ll 1$ case, the available experimental data [1] do not deal with sufficiently large Ra and Ta to allow a quantitatively meaningful comparison. By contrast, in the $\mathrm{Pr} \gg 1$ case, as we have seen from the derivation, in most of the region one has $\nu > \nu_t$ and the transfer only enters through the correction factor (20) while the bulk of (21) is mainly due to the external forcing f_i^e.

In conclusion, this work has made two predictions: one, for $\mathrm{Pr} > 1$, is verified by existing data, the second one, for $\mathrm{Pr} < 1$, will hopefully stimulate experimental and numerical simulation work to assess its validity (a paper with all the detailed derivations is being prepared).

The authors would like to thank Dr. R. E. Ecke for useful comments.

[1] H. T. Rossby, J. Fluid Mech. **36**, 309 (1969).
[2] F. Zhong, R. Ecke, and V. Steinberg, J. Fluid Mech. **249**, 135 (1993).
[3] Y. Liu and R. E. Ecke (to be published).
[4] K. Julien, S. Legg, J. McWilliams, and J. Werne, J. Fluid Mech. **322**, 243 (1996).
[5] G. A. Glatzmaier, J. Comput. Phys. **55**, 461 (1984).
[6] A. Tilger and F. H. Busse, J. Fluid Mech. (to be published).
[7] K. D. Squires, J. R. Chasnov, N. N. Mansur, and C. Cambon, in *Proceedings of the 74th Fluid Dynamics Symposium on Applications of LES and DNS to transition and Turbulence, Chania, Crete, Greece* (AGARD, Neuelly-sur-Seine, 1994).
[8] V. M. Canuto and M. S. Dubovikov, Phys. Rev. Lett. **78**, 666 (1997).
[9] V. M. Canuto and M. S. Dubovikov, Phys. Fluids **9**, 2118 (1997).
[10] C. Cambon, N. N. Mansur, and Godeferd, J. Fluid Mech. (to be published).
[11] V. M. Canuto, M. S. Dubovikov, and A. Dienstfrey, Phys. Rev. Lett. **78**, 662 (1997).
[12] V. M. Canuto, M. S. Dubovikov, and A. Dienstfrey, Phys. Fluids **9**, 2118 (1997).
[13] V. M. Canuto and M. S. Dubovikov, Phys. Fluids **8**, 571 (1996).
[14] V. M. Canuto and M. S. Dubovikov, Int. J. Mod. Phys. **12**, 3121 (1997).
[15] S. Chandrashekar, *Hydrodynamic and Hydromagnetic Stability* (Oxford University Press, London, 1961).
[16] E. Y. Kuo and M. C. Cross, Phys. Rev. E **47**, R2245 (1993).
[17] J. Herriman and F. H. Busse, J. Fluid Mech. **255**, 183 (1993).
[18] R. H. Kraichnan, J. Fluid Mech. **41**, 189 (1970).
[19] V. M. Canuto, M. S. Dubovikov, Y. Cheng, and A. Dienstfrey, Phys. Fluids **8**, 599 (1996).
[20] W. V. R. Malkus, Proc. R. Soc. Astron. **225**, 196 (1954).