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Orderly Spectra from Random Interactions

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We investigate the low-lying spectra of many-body systems with random two-body interactions, specifying that the ensemble be invariant under particle-hole conjugation. Surprisingly we find patterns reminiscent of more orderly interactions, such as a predominance of $J = 0$ ground states separated by a gap from the excited states, and evidence of phonon vibrations in the low-lying spectra. [S0031-9007(98)05710-X]

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In the spectra of molecules, atomic nuclei, and other many-body systems, the low-lying excitations often display a pattern suggestive of group symmetries, such as rotational or vibrational bands, even though the many-body spectrum is in principle complex and the interactions themselves have no trace of the symmetry groups displayed. This raises the question: To what extent does the low-lying spectrum acquire order simply from the most basic properties of the Hamiltonian? These properties include rotational invariance, possibly other symmetries such as isospin, and the fundamental nature of the interaction, which is predominantly two body in character. Given an ensemble of Hamiltonians of this form, some properties might occur often, while others would occur rarely and would depend sensitively upon the detailed form of the two-body interactions. An example might be a rotational spectrum: one could imagine that a typical ground state might behave as a solid. Then many members of the ensemble would have a rotational band built on the ground state. Stated another way, many-body calculations often rely upon model interactions, such as pseudopotentials in atomic and molecular physics, and the Skyrme, quadrupole-quadrupole, and other interactions in nuclear physics, that despite being drastic simplifications reproduce many key properties. We ask the logical extension: What properties remain as the Hamiltonian gets more and more arbitrary?

In this Letter, we begin exploratory studies of these questions, choosing ensembles of two-body random Hamiltonians and computing their many-body spectra. Although our own reference point is nuclear physics, we believe these issues may be relevant to generic many-body systems, such as molecules, atomic clusters, etc., and so our explorations should be considered in as broad an arena as possible. Obviously the choice of ensemble is crucial. In standard random matrix theory [1], a powerful principle for specifying the ensemble is to require that it be invariant under a change of basis. We shall use this principle at the level of the two-body Hamiltonian to construct our ensembles. We first choose a single-particle basis labeled by angular momentum j and two-particle states of good total angular momentum $J = [\, j \otimes j']$. States of the same angular momentum can be transformed into each other, so the ensemble is specified by the average of the matrix elements and their fluctuations for each J. For the symmetric matrix ensemble, invariant under orthogonal transformations, the mean square variance in matrix elements $V_{\alpha,\alpha'}$ is

$$\begin{aligned} \left\langle V_{\alpha,\alpha'}^2 \right\rangle &= c_J(1 + \delta_{\alpha\alpha'}), \\ \langle V_{\alpha,\alpha'} V_{\beta,\beta'} \rangle &= 0, \qquad (\alpha, \alpha') \neq (\beta, \beta'). \end{aligned} \tag{1}$$

Here α and α' label two-body states $|\, j \otimes j' = J\rangle$. Note that the variance depends only on J, and that there is the

usual factor of 2 difference between diagonal and off-diagonal matrix elements. The dependence of c_J on J will be relevant to determining the overall behavior of the ensemble. Obviously, pairing properties will dominate if $J = 0$ is enhanced. If the interaction is converted to a particle-hole representation, mean-field physics will become dominant if the diagonal $J = 0$ interaction is enhanced in that representation. For our study here, we follow the idea that the physics should be that of interacting quasiparticles, favoring neither a particle-particle nor a particle-hole representation. We therefore demand that the ensemble be invariant under the Pandya transformation [2],

$$\langle ij^{-1}; L|V|kl^{-1}; L\rangle = \sum_J (2J + 1) \begin{Bmatrix} j_i & j_l & J \\ j_k & j_j & L \end{Bmatrix} \times \langle il; J|V|kj; J\rangle .$$

The ensemble (1) is invariant under this transformation if and only if

$$c_J = \frac{\bar{v}^2}{2J + 1} . \tag{2}$$

Here $\bar{v}$ sets the energy scale for the ensemble, and our all results will be quoted in units of $\bar{v}$. Equations (1) and (2) define the ensemble to be studied in this Letter, which we term the *random quasiparticle ensemble* (RQE).

Random matrices were introduced into nuclear physics by Wigner [3] to model statistical properties of nuclear spectra. In particular the Gaussian orthogonal ensemble (GOE) of random Hamiltonians describes well the level repulsion found in distribution of nearest-neighbor spacings of states with the same quantum numbers. For more global properties, however, the GOE does not match real nuclei. The GOE gives a semicircle level density, while realistic shell-model Hamiltonians tend to give a Gaussian level density. But a GOE corresponds to Hamiltonians with interactions of all possible particle ranks, whereas shell-model Hamiltonians are only two-body interactions. Wong and French [4] investigated the *two-body random ensemble,* or TBRE (also sometimes termed the *embedded GOE*), which is similar to our RQE except that $c_J = \text{const}$. With the TBRE one regains Gaussian level densities and Mon and French [5] related the global level density to the moments of the ensemble. All these studies, however, considered only states with identical quantum numbers. In contrast, our work here examines the relation between states of different quantum numbers.

We stress that our Hamiltonians drawn from the RQE have no symmetries imposed on them beyond that of Eqs. (1) and (2) above. This is in contrast to earlier work [6,7] which studied linear combinations of a random Hamiltonian and a Hamiltonian containing a specified symmetry (e.g., SU(3) in [7]). These papers investigated the relative strength of the random Hamiltonian necessary to overwhelm the externally imposed symmetry. Rather than considering the interplay of a specified symmetry and a random Hamiltonian, we look to see what symmetries, or at least what markers of symmetries, can arise spontaneously in generic Hamiltonians.

We computed the low-lying spectra of random Hamiltonians for several different shell-model spaces. We label our systems by N, the number of identical particles, and Ω, the number of single-particle states. For the latter we consider two different single-particle spaces, first a space with j orbitals $\{\frac{1}{2}, \frac{3}{2}, \frac{5}{2}\}$, with $\Omega = 12$, and also in a space with j orbitals $\{\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}\}$, with $\Omega = 20$. In nuclear physics these correspond to the $1s_{1/2}$-$0d_{3/2}$-$0d_{5/2}$ and $1p_{1/2}$-$1p_{3/2}$-$0f_{5/2}$-$0f_{7/2}$ spaces, respectively. We considered $N = 6$ identical particles for both the $\Omega = 12$ and 20 spaces. A nuclear spectroscopist would identify these as ^{22}O and ^{46}Ca, respectively, but because our Hamiltonians have been significantly abstracted we prefer the abstract labeling scheme of $N = 6, \Omega = 12$ and $N = 6, \Omega = 20$.

In nuclear physics there is along with angular momentum an additional symmetry, isospin (which we remind our non-nuclear readers is an SU(2) symmetry between neutrons and protons and which is a nearly exact symmetry of the strong nuclear force). Since neutron-proton correlations might allow different statistical behavior, we enlarge the RQE to include isospin T which is treated exactly as J in our previous definition, so that $c_{J,T} = \bar{v}^2/(2J + 1)(2T + 1)$. For two-body interactions only the $T = 0, 1$ channels are possible. We studied the system with four protons and four neutrons (and thence $T_z = 0$) in the $\Omega = 12$ space, which corresponds to ^{24}Mg, but which we label as $N = 4, Z = 4, \Omega = 12$.

Before giving our results, we review the generic phenomenological features of the low-lying spectra of even N, even Z nuclides. In Nature, all even-even nuclei have $J = 0$ ground states which are pushed down in energy relative to the ground states of even-odd and odd-odd nuclei. The low-lying spectra display marked regularities, particularly in the spacing of the first $J = 0, 2, 4, 6, 8, \ldots$ states. One labels such regularities as "vibrational" or "rotational" bands depending if the excitation energy goes like J or $J(J + 1)$, respectively. Other regularities are also observed and associated with various group structures [8], but these are the most basic feature.

With this in mind, we now group our results under several major headings. We computed 1000 spectra for each system, with the Hamiltonians drawn from the RQE as defined in Eqs. (1) and (2). All single-particle energies were set to zero.

Predominance of $J = 0$ ground states.—For all our ensembles we found a predominance of $J = 0$ ground states. This is listed in Table I as a percentage. For the case with isospin, $N = 4, Z = 4, \Omega = 12$, we also required that the ground state have $T = 0$. (The other two cases with six identical particles automatically have $T = T_z = 3$.) We see that between two-thirds and three-quarters of the spectra have the singlet state as the lowest. This is not a trivial

TABLE I. Percentage of ground states (g.s.) of the RQE that have $J = 0, T = T_z$ for our target nuclides, as compared to the percentage of all states in the model spaces that have these quantum numbers.

N	Ω	Nucleus	$J = 0, T = T_z$ g.s.	$J = 0, T = T_z$ Total space
6	12	^{22}O	76%	9.8%
6	20	^{46}Ca	75%	3.5%
$N = 4, Z = 4$	12	^{24}Mg	66%	1.1%

consequence of the dimensionality of our model spaces, as may be seen in the last entry of Table I, showing the percentage of states in the model space that have the required quantum numbers. Furthermore, for the $N = 6, \Omega = 20$ case, there are considerably more $J = 2$ states than $J = 0$, 512 as compared to 137.

Gaps associated with $J = 0$ ground states.—In addition to a predominance of $J = 0$ ground states, such ground states are typically separated by a gap from the excited states. A typical case for $N = 6, \Omega = 20$ is shown in Fig. 1(a). Figure 2 shows the distribution of gaps for $J = 0$ ground states. The energy is in units of $\bar{v}$, the energy scale used in Eq. (2). In these units the centroids of the distributions are at $\sim\mathcal{O}(1)$, although with a broad width. For those ground states with $J \neq 0$ the gap is much smaller, as shown by an example in Fig. 1(b). For $N = 6, \Omega = 20$, the average energy gap between a $J = 0$ ground state and the first excited state is $0.47\bar{v}$ and for $N = 4, Z = 4, \Omega = 12$ it is $0.79\bar{v}$.

Vibrational/rotational "bands" and yrast structure.—In addition to the quantum numbers of the ground state, we investigated the evidence for band structure in the low-lying spectra. We characterize the low-lying $J = 0, 2, 4$ yrast states ("yrast" means the lowest state of a given angular momentum J) with energy E_J by the ratio $\rho = (E_4 - E_2)/(E_2 - E_0)$. If an interaction yields a vibrational spectrum, then $\rho = 1$, whereas a rotational spectrum gives $\rho = 7/3$. Shown in Fig. 3 is an analysis of the $J = 0, 2, 4$ spectrum for those samples in our ensemble which had a $J = 0$ ground state. All of our cases give broad peaks in the range $\rho = 0$ to ~ 1.

Note also that some interactions give a $J =$ 0-4-2 yrast character, as indicated by data at $\rho < 0$ in the figure. Of those samples that exhibit a $J = 0$ ground state, approximately 10% have a $J = 0, 2, 4$ spin ordering for the three lowest states.

Although there is no evidence of rotational collectivity among the first $J = 0, 2, 4$ states, the yrast spectrum extended to high angular momentum shows what is called "noncollective" rotational behavior in nuclear spectroscopy. This means that the energies of the yrast states E_J have an overall quadratic increase with J, but with large fluctuations from one J to the next. This is shown in Fig. 4, which displays averaged yrast spectra from our ensembles. When we fit $\langle E_J \rangle$ as a function of $J(J + 1)$ the long-range behavior is roughly linear with a slope of $(0.0539 \pm 0.0009)\bar{v}$.

Phonon collectivity.—In algebraic descriptions of collective behavior one sees far more than patterns in the excitation spectra: the low-lying states are connected to each other by operators that generate the group representation, or at least approximately so, depending on the

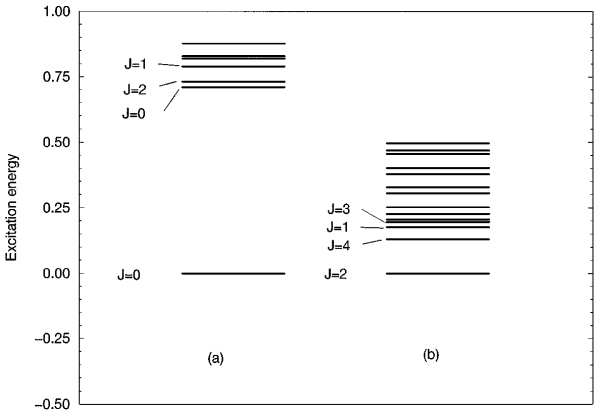

FIG. 1. "Typical" spectra for $N = 6, \Omega = 20$ (^{46}Ca) with an RQE Hamiltonian. Note the different ground state gaps for (a) ground state $J = 0$, (b) $J \neq 0$.

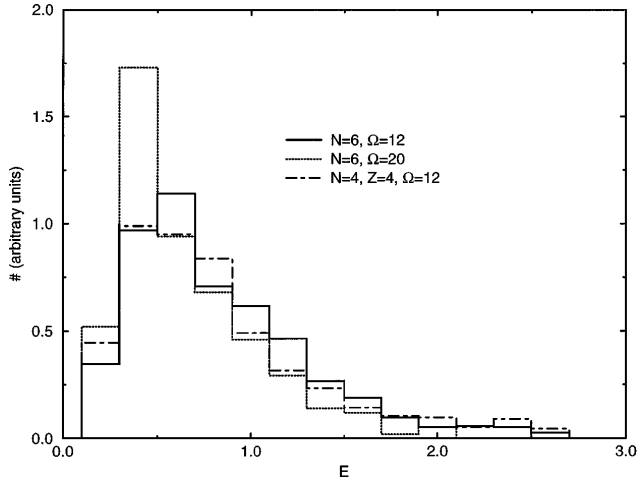

FIG. 2. Distribution of ground state gaps, defined as the excitation energy of the first excited state above a $J = 0$ ground state, in units of $\bar{v}$ [the energy scale from Eq. (2)].

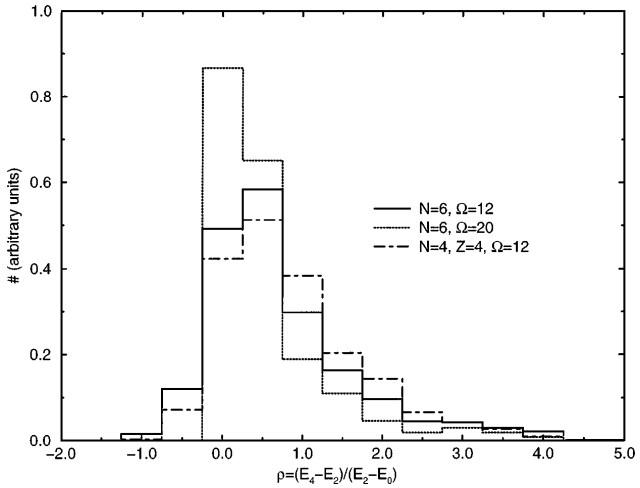

FIG. 3. Distribution of $\rho \equiv (E_4 - E_2)/(E_2 - E_0)$ for systems with $J = 0$ ground state. $\rho = 1$ for vibrational bands and $= 7/3$ for rotational bands.

goodness of the symmetry. These operators typically have a large component that is single particle in nature, i.e., expressible as a phonon: $\hat{X}^\dagger = \sum_{\alpha\beta} u_{\alpha\beta} a^\dagger_\alpha a_\beta$, where $a^\dagger, a$ are the usual fermion creation and annihilation operators, respectively. To see whether this collectivity carries over to the RQE we examined the transition between the ground $J = 0$ states (in the members of the ensemble that have such a ground state) and the first excited $J = 2$ state. For each member of the ensemble we define the phonon which maximally connects these two states by $u_{\alpha\beta} = \langle J = 2 | a^\dagger_\alpha a_\beta | J = 0 \rangle$. We then define the *fractional collectivity* f by the ratio of the strength of $\hat{X}^\dagger$ to the first excited state to the total strength of $\hat{X}^\dagger$ off the ground state,

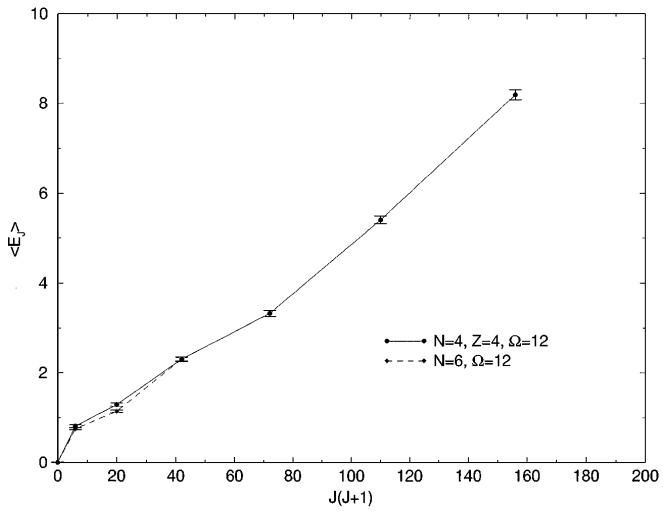

FIG. 4. Average excitation energy of yrast states (lowest state for a given J) as a function of $J(J + 1)$.

$$f = \frac{|\langle J | \hat{X}^\dagger | J = 0 \rangle|^2}{\langle J = 0 | \hat{X} \hat{X}^\dagger | J = 0 \rangle}. \quad (3)$$

If $f = 1$, then the excited state is completely described as a particle-hole excitation of the ground state. If f is very small then the two states are connected only by many-body operators.

We studied the fractional collectivity in the $N = 4, Z = 4, \Omega = 12$ system, and considered the particle-hole phonons between $J = 0$ ground states and the first $J = 2$ excited state. Averaging over our ensemble, we found that $\bar{f} = 0.52 \pm 0.27$. For comparison, the fractional collectivity using a realistic nuclear shell-model interaction [9], which is known to yield strong collectivity among these states, is 0.87. For totally random states (which we studied by computing the phonon between states generated from different interactions) $f \sim 10^{-2}$; this is what one would expect from the GOE. Therefore we find that the low-lying states of the RQE can be to a large degree related simply by particle-hole excitations.

In conclusion, the low-lying spectra of RQE Hamiltonians display markers of surprising regularity. Ground states are predominantly $J = 0$ and are pushed down relative to the rest of the spectrum, which are two of the important characteristics of the BCS pairing Hamiltonian. There is also evidence for low-lying vibrational states and noncollective rotational when averaged over a large number of yrast states, although no evidence for strong rotational collectivity. Perhaps we should not be surprised at these features of the RQE, as it was constructed with the Fermi liquid concept of quasiparticles in mind. The actual source of these apparent regularities is not evident, however.

The numerical studies presented here have barely scratched the surface of possible questions that can be addressed with two-body random ensembles. Although we have strong evidence for a pair gap, one would like to understand analytically whether this persists in the large-N limit and its functional dependence on the single-particle space. The choice of ensemble will also play a role in the physics. If the ensemble singles out the $J = 0$ interaction in the particle-hole channel, the Hamiltonian will favor mean-field physics, since the mean field is constructed out of the particle-hole density operators. Thus, we would expect predictions of mean-field physics, such as the Bethe level density formula, to emerge as a limit in this case. Another possibility is to emphasize pairing in the particle-hole channel. One would then expect to see phonons with more stability than in the RQE, and one could explore the more complex group structures that might arise (see Ref. [8]).

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