Noninterferometric Phase Imaging with Partially Coherent Light

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We demonstrate that interferometric imaging may be replaced by noninterferometric propagation-based techniques in many experiments. We explore propagation through the Poynting vector and find two classes of phase, one of which is topological in origin. Only this latter class may require interferometry to be determined, and even then only in specific well-defined circumstances. Our alternative definitions of phase are readily generalized to partially coherent radiation. Our analysis leads to an approach that is able to determine the absolute phase and the amplitude of a wave. [S0031-9007(98)05618-X]

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Interferometry is a key technique in the exploration of many aspects of physics and is the subject of continued development. Single crystal interferometers are now routinely used for x rays [1] and neutrons [2]. In recent years, the most intensive work has been the quest to construct an interferometer for atom optics [3]. All of this work is motivated by the sensitivity of phase based measurements in the context of both fundamental physics and for the measurement of physical properties.

Simultaneously, in adaptive optics there has been a substantial effort devoted to the development of rapid phase determination techniques [4]. This work has the aim of developing a sensor for atmospheric wave front distortions and building this sensor into an optical system able to correct the distortions in real time.

More recently, work has been published on the visualization of phase changes induced in very energetic x rays [5]; this work is of importance in the context of new third-generation x-ray sources. Related methods have allowed precise quantitative x-ray phase determination using noninterferometric propagation-based techniques [6] originating in adaptive optics research [7].

The key feature of interferometry is its ability to quantitatively measure the phase of the wave field. However, it is now apparent that noninterferometric techniques may also allow phase to be measured quantitatively and may offer a more stable and less technically demanding approach. The aim of this Letter is to explore the fundamental basis of this class of noninterferometric phase measurement techniques and to identify those classes of phase that are able to be determined without stringent coherence requirements. We find that, under conditions commonly implicit in interferometry, such as a uniform irradiance distribution, the propagation-based approaches are sensitive only to a class of phase we identify here as "scalar phase." Under these conditions interferometry is required to measure our "vector phase" component which is topological in origin and we conjecture to include the "Berry phase" [8] as a special case.

The formalism we develop allows the concept of phase to be generalized to include partially coherent radiation and so extends the range of applicability of phase determination techniques and substantially reduces the coherence requirements on the source.

Quantitative noninterferometric phase determination techniques have been based on the solution of the so-called transport-of-intensity equation [9], which relates the irradiance $I(\vec{r}_\perp, z)$ and phase $\phi(\vec{r}_\perp, z)$ of a paraxial monochromatic wave to its longitudinal irradiance derivative $\partial I(\vec{r}_\perp, z)/\partial z$. Note that where the paraxial approximation has been adopted, three-dimensional functions will be described in terms of the coordinate system $(\vec{r}_\perp, z)$, where z denotes position along the optical axis and $\vec{r}_\perp$ position within a plane normal to the optical axis. The transport of the intensity equation is derived by taking the imaginary part of an expanded form of the time-dependent paraxial wave equation $(\nabla_\perp^2 + 2ik\partial/\partial z)\sqrt{I(\vec{r}_\perp, z)}\exp[i\phi(\vec{r}_\perp, z)] = 0$, where $\nabla_\perp$ is the two-dimensional gradient operator acting in the plane containing $\vec{r}_\perp$, $k \equiv 2\pi/\lambda$, and λ is the wavelength. The resulting expression is [7,9]

$$-k\frac{\partial I(\vec{r}_\perp, 0)}{\partial z} = \nabla_\perp \cdot [I(\vec{r}_\perp, 0)\nabla_\perp \phi(\vec{r}_\perp, 0)]. \qquad (1)$$

Equation (1) relates the rate of change of irradiance in the direction of the optical axis to the irradiance and phase of the light in a plane perpendicular to the optical axis.

It has been shown that, in the absence of irradiance zeros, Eq. (1) has a unique solution for the phase [10]. Where the irradiance contains spatial variation, the equation is soluble, but the solution has previously appeared to be numerically difficult [11]; this issue is addressed in detail later in this paper. Particularly straightforward solutions are possible in the case of uniform irradiance and this has been demonstrated for hard x-ray phase imaging [6].

In this Letter we use the ideas underlying Eq. (1) to explore this approach to phase determination and show that it gives rise to a more general framework for phase and phase measurement.

Consider a complex free-space scalar wave $U(\vec{r}, t)$ associated with a general polychromatic field. As usual, we

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define the time-averaged irradiance of the field as $I_{\rm ave}(\vec{r}) = \langle |U(\vec{r},t)|^2 \rangle$, where $\langle\,\rangle$ denotes an average over time.

Poynting's theorem in free space, which embodies conservation of energy, may be written:

$$\nabla \cdot \vec{S}(\vec{r},t) + \frac{\partial}{\partial t} W(\vec{r},t) = 0\,, \tag{2}$$

where W is the appropriately defined energy density and $\vec{S}$ is the appropriately defined Poynting vector [12,13]:

$$\vec{S}(\vec{r},t) = -\frac{1}{8\pi}\Big[\nabla U(\vec{r},t)\frac{\partial}{\partial t}U^*(\vec{r},t) + \nabla U^*(\vec{r},t)\frac{\partial}{\partial t}U(\vec{r},t)\Big]. \tag{3}$$

Now consider a coherent scalar electromagnetic field of angular frequency ω:

$$U(\vec{r},t) \equiv \sqrt{I_{\rm coh}(\vec{r})}\exp[i(\phi(\vec{r}) - \omega t)]\,, \tag{4}$$

where $I_{\rm coh}(\vec{r})$ and $\phi(\vec{r})$ are, respectively, the irradiance and phase associated with the coherent field. The energy density W of a strictly monochromatic field is time invariant, and therefore the coherent version of (2) is

$$\nabla \cdot \vec{S}_{\rm coh}(\vec{r}) = 0\,, \tag{5}$$

where the subscript on the Poynting vector indicates a coherent field. Substitute (4) into expression (3) for the Poynting vector to obtain

$$\vec{S}_{\rm coh}(\vec{r}) = \frac{1}{4\pi} I_{\rm coh}(\vec{r})\omega\nabla\phi(\vec{r})\,, \tag{6}$$

so that Eq. (5) becomes

$$\nabla \cdot [I_{\rm coh}(\vec{r})\nabla\phi(\vec{r})] = 0\,. \tag{7}$$

This is identical to the coherent (nonparaxial) transport-of-intensity equation of Teague [9], although this equation first appears in the earlier paper of Green and Wolf [12].

We define the *normalized* Poynting vector as

$$\tilde{\vec{S}}(\vec{r}) \equiv \lim_{\epsilon\to 0^+} \vec{S}(\vec{r})/[I(\vec{r}) + \epsilon]\,. \tag{8}$$

If we are sufficiently removed from sources, the vector field $\vec{S}(\vec{r})$ is continuous and bounded. However, as $\epsilon \to 0$ the normalized vector field $\tilde{\vec{S}}(\vec{r})$ may be discontinuous across any zeroes of average irradiance that may be present.

Consider the coherent normalized Poynting vector, $\tilde{\vec{S}}_{\rm coh}(\vec{r})$. According to the Helmholtz decomposition theorem for vector fields, this field may be represented as the sum of the gradient of a scalar function $\phi_s(\vec{r})$, and the curl of a divergence-free vector function $\vec{\phi}_V(\vec{r})$:

$$\tilde{\vec{S}}_{\rm coh}(\vec{r}) = (\omega/4\pi)[\nabla\phi_S(\vec{r}) + \nabla \times \vec{\phi}_V(\vec{r})]\,. \tag{9}$$

We introduce the terms *scalar* and *vector phase* for the two functions ϕ_S and $\vec{\phi}_V$, respectively. They obey the differential equations:

$$\nabla^2\phi_S(\vec{r}) = \frac{4\pi}{\omega}[\nabla \cdot \tilde{\vec{S}}_{\rm coh}(\vec{r})]\,, \tag{10a}$$

$$\nabla^2\vec{\phi}_V(\vec{r}) = -\frac{4\pi}{\omega}[\nabla \times \tilde{\vec{S}}_{\rm coh}(\vec{r})]\,. \tag{10b}$$

Thus, these two functions are related to the Poynting vector and thereby to the phase of the wave via [14]

$$\phi_S(\vec{r}) = -\frac{1}{\omega}\int \frac{\nabla \cdot \tilde{\vec{S}}_{\rm coh}(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|}\,d^3\vec{r}\,' = -\frac{1}{4\pi}\int \frac{\nabla^2\phi(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|}\,d^3\vec{r}\,', \tag{11a}$$

$$\vec{\phi}_V(\vec{r}) = \frac{1}{\omega}\int \frac{\nabla \times \tilde{\vec{S}}_{\rm coh}(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|}\,d^3\vec{r}\,' = \frac{1}{4\pi}\int \frac{\nabla \times \nabla\phi(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|}\,d^3\vec{r}\,'. \tag{11b}$$

These are the fundamental quantities discussed in this paper.

Note that $\vec{\phi}_V(\vec{r}) = \vec{0}$, if $\nabla \times \nabla\phi(\vec{r}) = \vec{0}$. That is, $\vec{\phi}_V(\vec{r}) = \vec{0}$, if the phase of the coherent wave is single valued and continuous. The vector phase component is clearly topological in origin and we believe it is very closely related to the Berry phase [8]; this relationship will be explored in another paper.

With the notation introduced, the coherent Poynting theorem may be rewritten:

$$\nabla \cdot [I_{\rm coh}(\vec{r})\nabla\phi_S(\vec{r})] + \nabla I_{\rm coh}(\vec{r}) \cdot [\nabla \times \vec{\phi}_V(\vec{r})] = 0\,. \tag{12}$$

Thus, if the phase is continuous and single valued, then $\vec{\phi}_V(\vec{r}) = \vec{0}$, and we recover the standard transport-of-intensity equation where $\phi_S(\vec{r})$ reduces to the conventional phase of the optical field. This result identifies the nonuniqueness foreshadowed in earlier discussions of phase recovery using the transport-of-intensity equation approach [10].

If, however, $\vec{\phi}_V(\vec{r}) \neq \vec{0}$ then there are no measurable consequences to the flow of energy if the condition

$$\nabla I_{\rm coh}(\vec{r}) \cdot [\nabla \times \vec{\phi}_V(\vec{r})] = 0 \tag{13}$$

holds. If this condition is not obeyed, then there will, in general, be observable effects on the energy flow.

We conjecture that this discussion is related to the well-known Aharonov-Bohm effect [15], where quantum mechanical waves are passed perpendicular to a line of current. As the vector phase associated with such an experiment will be aligned with the line of current, Eq. (13) will be obeyed. Propagation-based phase determination will fail to detect the phase and so interferometry is needed to observe any phase shift.

Consider now a partially coherent field. In this case, the Poynting vector varies with time and the average energy flow is described by the time-averaged Poynting vector.

If the average Poynting vector field is normalized by the time-averaged irradiance to give $\tilde{\vec{S}}_{\rm ave}(\vec{r})$, we have a well-defined vector field which may be Helmholtz decomposed into the partially coherent analog of (9):

$$\tilde{\vec{S}}_{\rm ave}(\vec{r}) \equiv (\bar{\omega}/4\pi)[\nabla\phi_S(\vec{r}) + \nabla \times \vec{\phi}_V(\vec{r})], \quad (14)$$

where $\bar{\omega}$ is the mean angular frequency. The time-averaged form of Poynting's equation becomes

$$\nabla \cdot \vec{S}_{\rm ave}(\vec{r}) = \nabla \cdot [I_{\rm ave}(\vec{r})\nabla\phi_S(\vec{r})] + \nabla I_{\rm ave}(\vec{r}) \cdot [\nabla \times \vec{\phi}_V(\vec{r})] = 0, \quad (15)$$

which is identical to Eq. (12). Note that in this case there is no clear criterion for the existence of a nonzero $\vec{\phi}_V$; it may be shown that the vector phase may be present even where there are no irradiance zeros.

We see from Eq. (14) that the scalar and vector phases may be rigorously defined purely in terms of the normalized time-average Poynting vector via

$$\phi_S(\vec{r}) = -\frac{1}{\bar{\omega}}\int \frac{\nabla \cdot \tilde{\vec{S}}_{\rm ave}(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|}\, d^3\vec{r}\,', \quad (16a)$$

$$\vec{\phi}_V(\vec{r}) = \frac{1}{\bar{\omega}}\int \frac{\nabla \times \tilde{\vec{S}}_{\rm ave}(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|}\, d^3\vec{r}\,', \quad (16b)$$

and correspond to the previously defined phases [Eq. (11)] in the coherent limit. We thus have a generalized definition of phase that is appropriate in the context of propagation-based phase determination. For the remainder of this Letter we will drop any subscripts on the irradiance and Poynting vector in Eq. (15) as the expressions are valid whether the light is fully coherent or not.

We now turn to the problem of solving for the generalized phase of the wave field. It is straightforward to show that in the paraxial approximation Eq. (15) becomes

$$-\bar{k}\,\frac{\partial I(\vec{r}_\perp, 0)}{\partial z} = \nabla_\perp \cdot [I(\vec{r}_\perp, 0)\nabla_\perp\phi_S(\vec{r}_\perp, 0)] + \nabla_\perp I(\vec{r}_\perp, 0) \cdot [\nabla \times \phi_V(\vec{r})]_\perp, \quad (17)$$

where $\bar{k}$ is the appropriately averaged value for the wave number. In the absence of vector phases this is identical in form to the paraxial form of the coherent transport-of-intensity equation.

Consider the Helmholtz decomposition of the Poynting vector and let us discard the rotational component. Only in the case of uniform irradiance is this equivalent to assuming the absence of vector phase components. Under this condition we may write

$$\vec{S}(\vec{r}) = \nabla\psi(\vec{r}), \quad (18)$$

for some $\psi(\vec{r})$. The time-averaged Poynting theorem becomes

$$\nabla^2\psi(\vec{r}) = 0. \quad (19)$$

Let us adopt the paraxial approximation so that we may write

$$\nabla_\perp^2\psi(\vec{r}_\perp, 0) = -\frac{c\bar{k}^2}{4\pi}\,\frac{\partial I(\vec{r}_\perp, 0)}{\partial z}. \quad (20)$$

The phase may then be directly recovered using, for example,

$$\phi(\vec{r}_\perp, 0) = -\bar{k}\nabla_\perp^{-2}\left\{\nabla_\perp \cdot \left[\frac{1}{I(\vec{r}_\perp, 0)}\nabla_\perp\nabla_\perp^{-2}\frac{\partial I(\vec{r}_\perp, 0)}{\partial z}\right]\right\}, \quad (21)$$

where $\nabla_\perp^{-2}$ is the universe Laplacian calculated by whatever appropriate method. This reduces to the previously published form [6] in the case that the field is quasimonochromatic and the irradiance is uniform and given by I_0

$$\phi(\vec{r}, 0) = -k\,\frac{1}{I_0}\,\nabla_\perp^{-2}\,\frac{\partial I(\vec{r}, 0)}{\partial z}. \quad (22)$$

Thus, we have a practical approach for recovering both the amplitude and the phase, as defined here, of an arbitrary wave field. The solution is unique to within a phase component obeying Eq. (13).

To briefly demonstrate this approach to phase recovery we ran a simulation of the object in Fig. 1 being illuminated by a plane polychromatic x-ray wave. The image was taken to be 1 mm square and consisted of 256×256 pixels. The illumination was assumed to be in the x-ray band with a minimum wavelength of 0.6 Å and a maximum of 1.5 Å, and the irradiance reduced linearly from a maximum at 1.5 Å to 50% of this value at 0.6 Å. The phase object was taken to have the properties of carbon with a thickness varying from 0 to 50 μm. This produced a wavelength-dependent maximum phase shift of between 1.83π and 4.59π. Any absorption by the carbon was ignored for the purpose of the simulation. The absorption component of the object consisted of either 1 μm of gold or void. The transmission of the gold section thus varied from 71% to 88% over the spectrum. Any phase effects induced by the gold were ignored so as to provide a clear difference between the phase and the absorption components of the object. The wave field was numerically propagated to planes at a 5.0 cm distance on either side of the object. The longitudinal irradiance derivative was then estimated by simply taking the difference between the irradiance distributions on either side of the plane of interest, as would be done in an experiment. The phase and amplitude were then recovered via a fast-Fourier-transform implementation of Eq. (21), and took approximately one minute on an Intel Pentium Pro 200 MHz based personal computer. The results are shown in Fig. 2 and are identical, to within numerical accuracy, to the original phase/amplitude screen. Note that only the recovered phase is shown as the amplitude transmission is determined directly.

In conclusion, then, we have shown that propagation-based phase determination techniques may, in many cases, be used in place of interferometry and, therefore, may allow great simplification of experiments hitherto thought to require interferometry. Propagation-based approaches

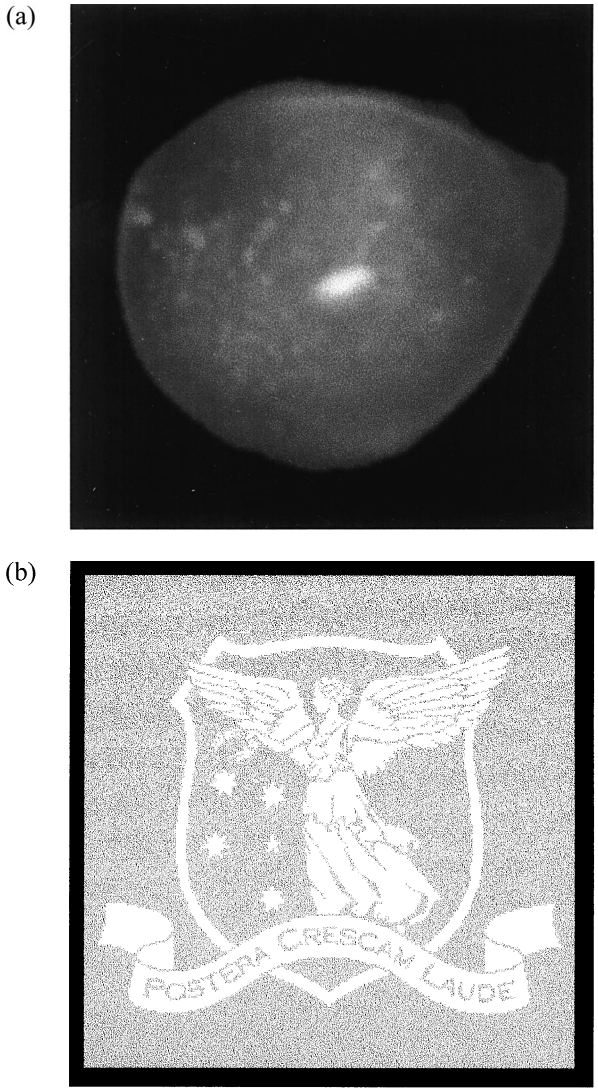

FIG. 1. Phase object thickness distribution (a) and average intensity transmission of radiation leaving the test object (b). The object is described in the text.

are far less restrictive on the coherence properties required and are able to determine all but a very restricted class of phase distributions; in the case of uniform irradiance this is the vector phase component introduced in our development. The scalar phase is equivalent to the conventional phase when the light is coherent and there are no wavefront singularities. We have also introduced a definition of the generalized phase of a light field which remains rigorously defined even when the radiation is partially coherent. We used these results to yield a quantitative noninterferometric partially coherent phase imaging algorithm.

The ideas developed here suggest that, in many circumstances, a perceived requirement for interferometry may need reexamination. Should interferometry not be required, then the results presented here may simplify some of the experiments in atom and neutron optics as the considerable difficulties faced in constructing an interferometry may be obviated by moving to a propagation-based phase determination approach. It is hoped, therefore, that this paper yields some useful insights for fundamental experiments of this form.

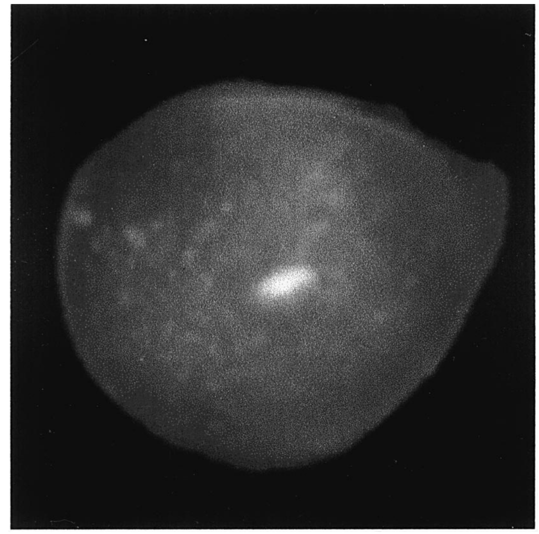

FIG. 2. Recovered thickness distribution using approach described in the text. It is indistinguishable from the input thickness variation.

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