

## Momentum Transformation Connecting a $NN$ Potential in the Nonrelativistic and Relativistic Two-Nucleon Schrödinger Equation

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An analytical relation between center of mass momenta in a nonrelativistic and a relativistic two-nucleon Schrödinger equation is proposed which allows analytically rewriting the two Schrödinger equations into each other. As a consequence a  $NN$  potential occurring in the relativistic Schrödinger equation can be gained from a nonrelativistic one by an analytical procedure. The  $S$  matrices in the two equations are exactly identical, and therefore the two-nucleon phase shifts are also. [S0031-9007(98)05586-0]

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Few-nucleon equations and their numerical solutions are mostly carried out in a nonrelativistic framework. Fitting  $NN$  potentials to  $NN$  data incorporates to some extent relativistic features into the nonrelativistic framework; in other words, relativistic features contained in the data are absorbed in the potential parameters of the nonrelativistic Schrödinger equation. This, however, is not sufficient; especially relativistic effects in systems with more than two particles are not accessible in this manner. As a first step towards a relativistic framework the  $NN$  system in its center of mass (c.m.) frame should have the correct form [1,2], which requires that the operator for the kinetic energy is formed out of square roots. The present day so-called realistic  $NN$  forces are fitted to  $NN$  data together with the nonrelativistic form of the kinetic energy [3–5]. More precisely a mixed procedure has been applied. These potential models were fitted to the phase shifts and amplitudes of the Nijmegen partial-wave analysis [6], whereby relativistic relations between the c.m. momenta and the kinetic energy and between the differential cross section and the scattering amplitude have been used. Clearly consistent steps would be desirable, as well as refraining altogether from using the nonrelativistic  $NN$  Schrödinger equation. The question addressed in this paper is whether the potentials have to be refitted to the data once the relativistic form of the kinetic energy is used instead of the nonrelativistic one. A refitting of the  $NN$  potential has been undertaken, e.g., in [7]. In earlier work [8,9] approximate relations among the potentials in the nonrelativistic and relativistic Schrödinger equations were introduced. In Ref. [10] relations among the squares of the interacting and the free mass operators are used to find formal relations between the relativistic and nonrelativistic Schrödinger equation. However, these formal relations do not allow one to write the potential in the relativistic Schrödinger equation explicitly in terms of the potential in the nonrelativistic Schrödinger equation. We want to propose here a procedure providing an exact analytic relation between those potentials. Before, however, we would like to emphasize that we are not generating a  $NN$  potential which has all the

correct relativistic features built in (like an effective potential derived from field theory in the Hamiltonian formalism [11]) but simply establishing a formal connection among potentials in the relativistic and nonrelativistic forms of the Schrödinger equation. The potentials are treated as “black boxes” simply as functions of certain momenta with no reference to whether the potentials originally had some relativistic background or not.

In practice few-nucleon problems are solved mostly in a partial-wave representation. Therefore we formulate that transformation among the potentials for given angular momentum states. Let  $l$ ,  $s$ , and  $j$  denote the orbital, total spin, and total angular momenta of the  $NN$  system, then for a given  $j$  and  $s$  the nonrelativistic Schrödinger equation in momentum space reads

$$\left(\frac{p^2}{m} + 2m\right)\psi_l(p) + \sum_{l'} \int_0^\infty dp' p' \times p'^2 V_{ll'}(p, p') \psi_{l'}(p') = E \psi_l(p). \quad (1)$$

Here  $m$  is the nucleon mass and the sum over  $l'$  is present or not depending on  $s$  and  $j$ . In Eq. (1) the rest masses  $2m$ , which are usually absorbed into the definition of the energy, are still contained. The relativistic version of the two-nucleon Schrödinger equation is given by [1,2,10]

$$2\sqrt{m^2 + p^2} \phi_l(p) + \sum_{l'} \int_0^\infty dp' p' \times p'^2 U_{ll'}(p, p') \phi_{l'}(p') = E \phi_l(p). \quad (2)$$

The question is whether one can find the potential  $U$  in Eq. (2) once the potential  $V$  in the nonrelativistic Schrödinger equation (1) is given.

We propose the use of two types of momenta and denote the momenta in the nonrelativistic Eq. (1) by  $q$ . We define the following relation between the momenta  $q$  in Eq. (1) and  $p$  in Eq. (2):

$$2\sqrt{m^2 + p^2} \equiv 2m + \frac{q^2}{m}. \quad (3)$$

This relation can be solved either for  $p$  or for  $q$ ,

$$p = q \sqrt{1 + \frac{q^2}{4m^2}} \quad (4)$$

and

$$q = \sqrt{2m} \sqrt{E_p - m}, \quad (5)$$

where we define  $E_p = \sqrt{m^2 + p^2}$ . For small values of  $p$  or  $q$  one obtains

$$p \approx q \left( 1 + \frac{q^2}{8m^2} \right) \quad (6)$$

and

$$q \approx p \left( 1 - \frac{p^2}{8m^2} \right), \quad (7)$$

whereas for large values of  $p$  and  $q$  one gets

$$p \approx \frac{q^2}{2m}. \quad (8)$$

With these definitions it is possible to rewrite Eq. (1) into Eq. (2) and vice versa. Using Eqs. (3) and (4) one obtains

$$\begin{aligned} p^2 dp &= q^2 dq \sqrt{1 + \frac{q^2}{4m^2}} \left( 1 + \frac{q^2}{2m^2} \right) \\ &\equiv q^2 dq h^2(q), \end{aligned} \quad (9)$$

with

$$h(q) \equiv \sqrt{\left( 1 + \frac{q^2}{2m^2} \right) \sqrt{1 + \frac{q^2}{4m^2}}}. \quad (10)$$

Then Eq. (2) turns into

$$\left( 2m + \frac{q^2}{m} \right) \phi_l(p) + \sum_{l'} \int_0^\infty dq' \times q'^2 h^2(q') U_{ll'}(p, p') \phi_{l'}(p') = E \phi_l(p). \quad (11)$$

If we define

$$\psi_l(q) \equiv h(q) \phi_l(p) \quad (12)$$

and

$$V_{ll'}(q, q') \equiv h(q) U_{ll'}(p, p') h(q'), \quad (13)$$

we arrive at

$$\left( 2m + \frac{q^2}{m} \right) \psi_l(q) + \sum_{l'} \int_0^\infty dq' \times q'^2 V_{ll'}(q, q') \psi_{l'}(q') = E \psi_l(q), \quad (14)$$

which is the nonrelativistic Eq. (1). Thus Eqs. (12) and (13) provide the desired relations between the wave functions and the potentials. Explicitly written, the potential  $U$  results from  $V$  via

$$\begin{aligned} U_{ll'}(p, p') &= h^{-1}(q) V_{ll'}(q, q') h^{-1}(q') \\ &= \frac{\sqrt{2} m}{\sqrt{E_p} \sqrt{2m^2 + 2mE_p}} \\ &\times V_{ll'}(\sqrt{2m} \sqrt{E_p - m}, \sqrt{2m} \sqrt{E_{p'} - m}) \\ &\times \frac{\sqrt{2} m}{\sqrt{E_{p'}} \sqrt{2m^2 + 2mE_{p'}}}, \end{aligned} \quad (15)$$

and the potential  $V$  results from  $U$  via

$$\begin{aligned} V_{ll'}(q, q') &= \sqrt{\left( 1 + \frac{q^2}{2m^2} \right) \sqrt{1 + \frac{q^2}{4m^2}}} \\ &\times U_{ll'} \left( \sqrt{q^2 + \frac{q^4}{4m^2}}, \sqrt{q'^2 + \frac{q'^4}{4m^2}} \right) \\ &\times \sqrt{\left( 1 + \frac{q'^2}{2m^2} \right) \sqrt{1 + \frac{q'^2}{4m^2}}}. \end{aligned} \quad (16)$$

Correspondingly, the relativistic wave function expressed in terms of the nonrelativistic one is given by

$$\phi_l(p) = \frac{\sqrt{2} m}{\sqrt{2m^2 + 2mE_p} \sqrt{E_p}} \psi_l(\sqrt{2m} \sqrt{E_p - m}). \quad (17)$$

The transformation of the wave function given in Eq. (12) conserves the norm. We have

$$\begin{aligned} \int_0^\infty dp p^2 \phi_l^2(p) &= \int_0^\infty dq q^2 h^2(q) \phi_l^2(p) \\ &= \int_0^\infty dq q^2 \psi_l^2(q). \end{aligned} \quad (18)$$

Let us now consider the Lippmann-Schwinger equations for the partial-wave projected half-shell  $t$  matrices. The relativistic version is given by

$$\begin{aligned} T_{ll'}(p, p') &= U_{ll'}(p, p') + \sum_{l''} \int_0^\infty dp'' p''^2 U_{ll''}(p, p'') \\ &\times \frac{1}{2E_{p'} - 2E_{p''} + i\epsilon} T_{l''l'}(p'', p'). \end{aligned} \quad (19)$$

Using Eqs. (9) and (13) and defining

$$t_{ll'}(q, q') = h(q) T_{ll'}(p, p') h(q'), \quad (20)$$

we obtain from Eq. (19)

$$\begin{aligned} t_{ll'}(q, q') &= V_{ll'}(q, q') + \sum_{l''} \int_0^\infty dq'' q''^2 V_{ll''}(q, q'') \\ &\times \frac{1}{\frac{q'^2}{m} - \frac{q''^2}{m} + i\epsilon} t_{l''l'}(q'', q'). \end{aligned} \quad (21)$$

This is the standard nonrelativistic Lippmann-Schwinger equation.

Equation (20) also provides a relation between the phase shifts evaluated through the relativistic and nonrelativistic Lippmann-Schwinger equations. The unitarity relations resulting from Eqs. (19) and (21) are given by

$$\text{Im } T_{ll}(p, p) = -\frac{\pi p E_p}{2} \sum_{l'} |T_{ll'}(p, p)|^2 \quad (22)$$

and

$$\text{Im } t_{ll}(q, q) = -\frac{\pi q m}{2} \sum_{l'} |t_{ll'}(q, q)|^2. \quad (23)$$

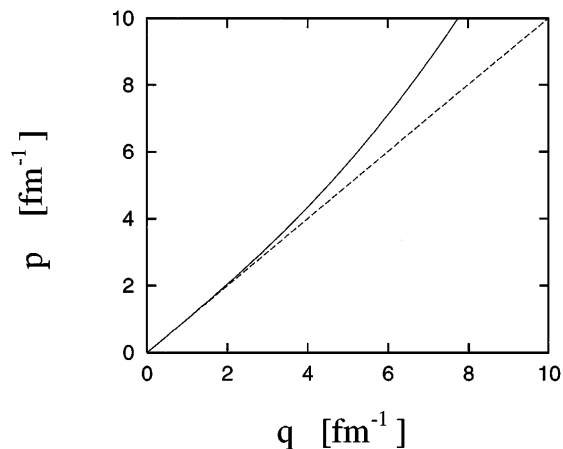


FIG. 1. The relation between the relativistic momentum  $p$  and the nonrelativistic momentum  $q$  as given in Eq. (4).

As a consequence the  $S$  matrices given by

$$S_{ll'}(p) \equiv \delta_{ll'} - i\pi p E_p T_{ll'}(p, p) \quad (24)$$

and

$$s_{ll'}(q) \equiv \delta_{ll'} - i\pi q m t_{ll'}(q, q) \quad (25)$$

are unitary. Using Eq. (20) and the relations (3) and (4) one obtains

$$\begin{aligned} s_{ll'}(q) &= \delta_{ll'} - i\pi q m h(q) T_{ll'}(p, p) h(q) \\ &= \delta_{ll'} - i\pi q m \sqrt{1 + \frac{q^2}{4m^2}} \left(1 + \frac{q^2}{2m^2}\right) T_{ll'}(p, p) \\ &= \delta_{ll'} - i\pi q m \frac{p}{q} \frac{1}{2m} \left(2m + \frac{q^2}{m}\right) T_{ll'}(p, p) \\ &= \delta_{ll'} - i\pi p E_p T_{ll'}(p, p) = S_{ll'}(p). \end{aligned} \quad (26)$$

From Eq. (26) we see that the  $S$  matrices are identical, and as a consequence the scattering phase shifts and mixing parameters are identical. Though the momenta  $q$  and  $p$  for the nonrelativistic and relativistic  $S$  matrices are different, the related energies are the same, namely,  $E = 2m + q^2/m = 2\sqrt{m^2 + p^2}$ .

In Fig. 1 we show the relation between the momenta  $p$  and  $q$  as given in Eq. (4). In Fig. 2 we display the  $s$ - and  $d$ -wave components of the deuteron wave function. We show these wave function components as a function of their respective momenta,  $p$  in the relativistic case and  $q$  in the nonrelativistic case. The underlying  $NN$  potential was arbitrarily chosen as the charge dependent (CD) Bonn potential [3]. The minima for the  $s$ - and  $d$ -wave components are shifted towards larger momenta in the relativistic case. The effects are large above about  $5 \text{ fm}^{-1}$ .

Summarizing, we have shown that  $NN$  potentials fitted to  $NN$  data in a nonrelativistic framework can be analytically rewritten by a scale transformation in the momenta such that they lead to the same  $S$  matrix when used in a relativistic Schrödinger equation. There is no need for

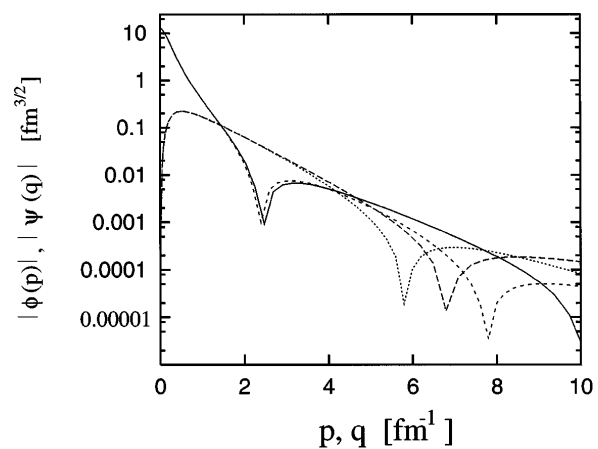


FIG. 2. The absolute values of the relativistic and nonrelativistic wave functions,  $\phi(p)$  and  $\psi(q)$  for the deuteron bound state. The  $s$ -wave components  $\phi_0(p)$  (solid curve) and  $\psi_0(q)$  (short dashed curve) and the  $d$ -wave components  $\phi_2(p)$  (long dashed curve) and  $\psi_2(q)$  (dotted curve) have large differences at higher momenta.

refitting parameters when the nonrelativistic operator of kinetic energy is replaced by the relativistic square root expressions.

The potential  $U$  can be used in few-nucleon systems with  $A > 2$  as done in the studies [7,8].

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