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Role of Coincidental Nonlinear Events in the Saturation of Moderately Damped Modes

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Both the level of broadband microturbulence and the behavior of driven waves can be strongly affected by local coincidences that produce nonlinear levels of the total fluctuation amplitude. We evaluate the rate at which such coincidences occur, for moderately damped modes which frequently reset their phases. They can produce significant nonlinear effects and can limit the amplitude of individual modes to low levels. These effects, which involve very many modes, are inherently difficult to observe using standard Fourier space or particle simulation techniques. [S0031-9007(98)05658-0]

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Given that particulate media sustain many modes of oscillation, it is worth considering the nonlinear effects that may arise from the interaction of these modes. When one or a few isolated modes are intense, then phenomena such as parametric instabilities or multiwave mixing are likely. However, large, local, nonlinear effects are possible even in cases for which the oscillation amplitude of each mode is small. If, in any given region, enough of the modes are in phase, then the total fluctuation amplitude may become large enough that nonlinearities become strong. This may not be important for modes which are so weakly damped that they sample the entire system and maintain their phase relationships for long periods. However, physical systems often also include modes which are moderately damped, so that a given wave packet endures for many wave cycles yet damps before it samples much of the system. Such modes reset their phases frequently, through emission and absorption. From time to time, their phases may be such that a strong nonlinear interaction occurs. The natural question to ask is as follows: Whether and when are such nonlinear interactions important? In the present Letter, we take a statistical approach to the analysis of this question. We determine the conditions under which the nonlinear interaction of many weak modes may be significant, and show that these conditions may indeed occur in some physical systems.

Key historical work on the problem of oscillations in many-body systems is due to Lord Rayleigh [1]. A system of many particles, with three degrees of freedom and negligible damping, is well known to sustain $3N$ normal modes. As was realized by Rayleigh [1], and as has been further explored in more recent work [2,3], the presence of weak damping causes some of these modes to be coupled and may or may not prevent the existence of classical normal modes. Current work has begun to go beyond a pure normal-mode analysis in areas such as mechanical structures [4] and atmospheric acoustics [5,6]. The author has been able to find no work which studied the problem of modes which are moderately damped in the sense described above. Yet such a situation does occur in real systems. Acoustic modes, in particular, may be damped on scales of interest, as they are in the ocean, in plasmas, in football stadiums, and in solid-state acoustics [7].

Moreover, any effects which arise from the occasional nonlinear interactions of large numbers of such modes may not be observed in most, if not all, current computational simulations [8]. Because such simulations, of necessity, discretize their physical system using far fewer grid cells or macroparticles than are actually present in the physical system, they sustain far fewer modes of oscillation than natural systems. They will miss any physical effects which require more modes than they can sustain. Likewise, the analysis of wave behavior in a medium, through Fourier transforms, is likely to miss such local interactions, since, in Fourier space, they represent a coupling across many frequencies and wave numbers. These observations motivate the statistical approach taken here.

We consider a system in which there is a very large number of moderately damped modes. The total fluctuation amplitude at any given position is usually the sum of the amplitudes of all of the modes present. From time to time, however, a sufficient fraction of the modes may be in phase, at some specific location, to produce to coincidental nonlinear event (CNE). We define such an event as the production of a state in which the total fluctuation amplitude is so large that the response of the medium becomes strongly nonlinear. We will analyze this problem in terms of a density of particles, but other variables such as electric field could be used equally well. We define the minimum fluctuation amplitude for which there is such a response to be $\tilde{n}_{\mathrm{nl}}$, which is normalized to the average density n so that the maximum possible value of $\tilde{n}_{\mathrm{nl}}$ is 1. In gases, $\tilde{n}_{\mathrm{nl}}$ will be near 1; in plasmas, it probably exceeds 0.1; and in solids or liquids, it will be much smaller. In response to such a very large local fluctuation amplitude, one expects that much of the energy from the individual local modes will be nonlinearly absorbed.

In an actual medium, for modes whose linear damping is large enough that a typical local oscillation is damped before it reaches the system boundary, the modes can be viewed as a collection of wave packets whose shape is determined by their origin and by their damping. These wave packets are randomly created, by whatever mechanism, and interact when they overlap. Given a thorough understanding of the creation and damping mechanisms of such modes, one could, in principle, produce a detailed model of their properties and interactions for any specific case. This might prove difficult in some cases, however, because the relevant number of modes can easily exceed 10^5 and powers thereof. We can instead turn this large number of modes to our advantage, as they permit us to adopt a statistical description based upon their characteristic properties.

In order to obtain a statistical estimate of the rate of CNEs, we need to identify the number of wave packets at a given location with statistically independent phases. We approximate the wave packets as Gaussian wave packets. The phases of these wave packets are randomly reset after one characteristic linear damping time of a typical mode (or after each CNE), because the mode at a given characteristic frequency must be reemitted, giving it a new phase, on such a time scale. Thus, the normalized fluctuation amplitude $\delta n/n$ of a wave packet of central frequency ω_1, linear damping δ, and phase ϕ has a frequency spectrum given by

$$\frac{\delta n}{n} = \tilde{n} e^{i\phi} \exp\left[-\frac{1}{2} \frac{(\omega - \omega_1)^2}{\delta^2}\right]. \tag{1}$$

The FWHM of such wave packets is quite close to 2δ, which we take to be the mode spacing in frequency space. We note that, while this procedure provides a means of counting the number of independent modes in phase space, the actual wave packets are not spaced by just this amount. In actual physical systems, wave packets can be emitted having arbitrary central frequencies ω_1. When two such wave packets are sufficiently close in frequency, in $\mathbf{k}$, in time, and in space, as determined by linear damping rates, they become effectively indistinguishable, which is the justification for counting wave packets as is done here.

To identify the moderately damped modes, of interest here, we divide the phase space of the wave packets into regions, based on their physical properties. Thus, for example, we divide longitudinal modes into long wavelength modes, which sample the entire system, acoustic modes, which will serve as our example here, and short-wavelength modes (such as optical phonons or the equivalent), which have different properties. For the purpose of evaluating the effects of broadband turbulence, we will approximate the modes within a given region of phase space as having the same amplitude.

The number of independent wave packets at some location includes all of the packets which can reach that location without being substantially damped. We assume here that the medium has an isotropic dielectric function. In this case, the local volume V_{loc} from which such wave packets originate is $4\pi R^3/3$, where R is a characteristic linear damping distance. It is clear that R is proportional to $1/\delta$. By defining $R = (\pi/\delta)d\omega/dk$, we establish the same mode spacing in k space and ω space. The number of modes in an element of wave number space, $d\mathbf{k}$ is then $(V_{\mathrm{loc}}/8\pi^3)\,d\mathbf{k}$.

The net fluctuation amplitude, at some location, is the sum of the instantaneous amplitudes of all of the wave packets present at that location. The number of such packets is

$$N = \int \frac{V_{\mathrm{loc}}}{8\pi^3}\,d\mathbf{k} = \int \frac{\pi}{6\delta^3}\left(\frac{d\omega}{dk}\right)^3 d\mathbf{k} = \int \frac{2\pi^2}{3\delta^3}\left(\frac{d\omega}{dk}\right)^3 k^2\,dk\,, \tag{2}$$

where the final equality applies only to isotropic turbulence and the integral is over the relevant range of wave vectors. This integral may or may not be simple. For example, $d\omega/dk$ may equal the sound speed c_s and δ may be proportional to the frequency $\delta = \alpha c_s k$. In this case,

$$N = \frac{2\pi^2}{3\alpha^3}\ln(k_{\max}/k_{\min})\,, \tag{3}$$

where $k_{\min}$ and $k_{\max}$ are the minimum and maximum wave numbers in the region over which the broadband turbulence has similar properties.

We would like to know how often these N modes are phased to produce a CNE. In principle, one could consider an ensemble of all possible distributions of the phases of the N modes. Here, we use a simpler estimate. We divide these N packets into four groups identified as n_1, n_2, n_3, and n_4. These modes are those with phases within $\pm\pi/4$ of 0, $\pi/2$, π, and $3\pi/2$, respectively, at the time when the largest total fluctuation amplitude is produced for a given realization of the phases. To a first approximation, groups 1 and 3 are phased so that they do not interfere with groups 2 and 4. Given a constant mode amplitude $\tilde{n}$ for broadband turbulence, this maximum fluctuation amplitude is

$$\tilde{n}_{\max} = \tilde{n}|n_1 - n_3|. \quad (4)$$

The average value of $(n_1 - n_3)$ is 0, but there is a definite probability of finding any value up to $\pm N$. We define $\Delta = n_1 - n_3$ and $d = n_1 + n_3 - N/2$. By application of the usual statistics of large systems, one can show that the probability distribution for a state having Δ and d is

$$W(\Delta, d) = \frac{\exp\{-\Delta^2/[2(d + N/2)]\}}{\sqrt{2\pi(d + N/2)}} \frac{\exp[-d^2/(N/2)]}{\sqrt{\pi N/2}}. \quad (5)$$

We want the probability distribution $W(\Delta)$, which we obtain by integrating Eq. (5) over d. [This is straightforward after realizing that $W(\Delta, d)$ is significant only for $d \ll N/2$.] This yields

$$W(\Delta) = \frac{1}{\sqrt{\pi N}} \exp\left[-\frac{\Delta^2}{N}\right]. \quad (6)$$

The probability that the difference Δ will produce a CNE is the integral of W over all Δ above the minimum value where this is the case, $\Delta_0 = \tilde{n}_{\rm nl}/\tilde{n}$. Allowing for both positive and negative Δ, this integral is the complementary error function $\mathrm{erfc}(u_0)$, where $u_0 = \Delta_0/\sqrt{N}$. (The integral can be taken to infinity with negligible error.) The statistical imbalance of groups 2 and 4 contributes an equal probability of a CNE. The total probability that a given distribution of phases will produce a CNE, from broadband turbulence whose normalized level is $\tilde{n}\sqrt{N}/\tilde{n}_{\rm nl} = 1/u_0$, is then

$$P(u_0) = 2\,\mathrm{erfc}(u_0) - \mathrm{erfc}^2(u_0). \quad (7)$$

Figure 1 shows the dependence of P on the normalized level of turbulence. One sees that a small change in $\tilde{n}$ can produce a large change in P and that the importance of CNEs increases sharply as $\tilde{n}$ increases. It is not surprising that the probability of producing a CNE becomes substantial as $\tilde{n}\sqrt{N}$ approaches $\tilde{n}_{\rm nl}$. One can also see, here, the difficulty that simulations will have in observing such phenomena. Simulations reduce $\sqrt{N}$ without being able to reduce $\tilde{n}_{\rm nl}$, and, thus, may require inherently nonlinear levels of $\tilde{n}$ before CNEs will appear.

We can now identify the conditions under which CNEs may produce significant observable effects. Once $\tilde{n}\sqrt{N} \sim \tilde{n}_{\rm nl}$, one is likely to see evidence of them in observations which are sensitive to the occurrence of nonlinear damping events. These might include the observation of energetic particles in plasmas or the appearance of bursts of noise in other frequency bands in condensed media. A next question is as follows: Can the damping due to CNEs ever become large enough to impact the turbulence itself?

The rate at which a CNE occurs locally depends on the probability P that a given ensemble of wave packets will produce one and on the rate Γ at which new ensembles are established. In the absence of a CNE, a new, independent distribution of phases is established at the characteristic linear damping rate δ. After a CNE, both the phases and the amplitudes must be reestablished. The phases will be disrupted by the very rapid amplitude changes produced by the damping event, and will establish new values following the event at roughly the acoustic rate ω. The rate at which the amplitudes are reestablished, ν, will depend upon the type and amplitude of the driving source. Cherenkov sources will be effective on particle transit time scales. Mode coupling sources [9] from waves at other frequencies will have rates that depend on their strength but that can easily exceed δ. (Note that the physical system under consideration must have local sources—the moderately damped waves cannot propagate to a given location from beyond its local interaction volume.) The combined effect of these two alternatives is that $\Gamma = \delta(1 - P) + \nu P$, so that the rate at which a CNE occurs locally is $\Gamma P = \delta P + (\nu - \delta)P^2$.

We now apply this to the damping of the modes themselves, which is due to the linear damping at rate δ, and to the effects of CNEs, which occur at rate ΓP. We assume here that a CNE, through its nonlinear interaction, removes most of the energy from the modes under consideration in $V_{\rm loc}$. The energy loss due to CNEs will be comparable to or larger than the linear damping when $\nu P^2 > \delta$. In combination with Eq. (7), this implies that CNEs will be a major energy sink when

$$\mathrm{erfc}(u_0) = \left[1 - \sqrt{1 - \sqrt{\delta/\nu}}\right]. \quad (8)$$

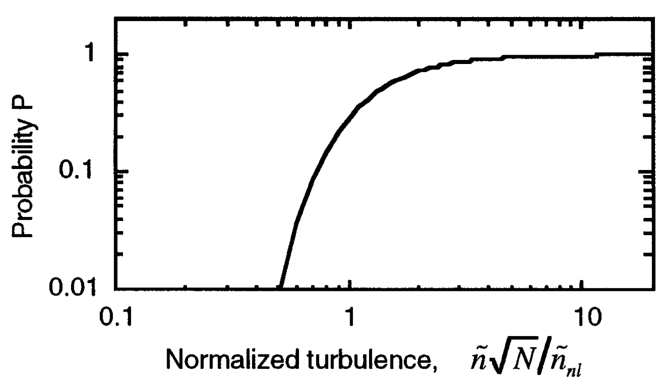

FIG. 1. The probability P that a random set of phases will produce a coincidental nonlinear amplitude is very strongly dependent upon the normalized level of broadband turbulence.

Figure 2 shows this result. We see that, once the modes are driven strongly enough to increase their amplitude above $\tilde{n} = \tilde{n}_{\rm nl}/\sqrt{N}$ in a small fraction of a damping time, the coincidental nonlinear events will dominate the energy losses.

This specific analysis implies that CNEs impose a fairly small limit on the general level of broadband turbulence that can be driven without a large increase in the driving power and corresponding heating of the medium. In gases and plasmas, where the $\tilde{n}_{\rm nl} \sim 1$, this implies that the broadband turbulence cannot exceed $\tilde{n} \sim 1/\sqrt{N}$. This can be far below the level at which wave-wave or wave-particle nonlinearities, involving only one or a few waves, become strong.

We can take a numerical example from the laser-plasma context. We consider acoustic waves for which $\omega = c_s k$, $d\omega/dk = c_s$, and $\delta \sim 0.03\omega = 0.03 c_s k$ so that $\alpha = 0.03$. The region of interest extends from $k_{\rm min} \sim 3000\ {\rm cm}^{-1}$, as waves having smaller wave numbers are too weakly damped to have consequences on the time scales of interest, to $k_{\rm max} \sim 10^6$, as waves with larger wave numbers have $k\lambda_D > 1$ and, thus, are strongly damped. (Here, λ_D is the Debye shielding distance.) Equation (3) then implies $N \sim 10^6$. The maximum amplitude of broadband turbulence permitted in such a plasma is $\tilde{n} \sim 1/\sqrt{N} \sim 0.001$. This would imply a reflectivity of $\sim 0.01\%$ from a plasma whose density is 0.05 times the critical density, taking the coherence length to be R. This is roughly consistent with observations in some laser plasmas, where the level of acoustic turbulence is apparently quite large [10–16].

The analysis presented here has considered broadband turbulence as an initial, tractable problem for evaluation. The fundamental result is that the coincidental overlap of many modes can produce nonlinear effects even when the mode amplitudes are very small by single-mode measures. This result will apply as well to structured turbulence, for which the quantitative analysis will be more complex. Ideally, one would carry out a detailed evaluation or simulation of the statistical properties of the total fluctuation amplitude produced by the entire, actual spectrum of modes. In practice, it may prove more feasible to treat the structure as a number of discrete intervals of uniform turbulence, to which the present analysis can be directly extended.

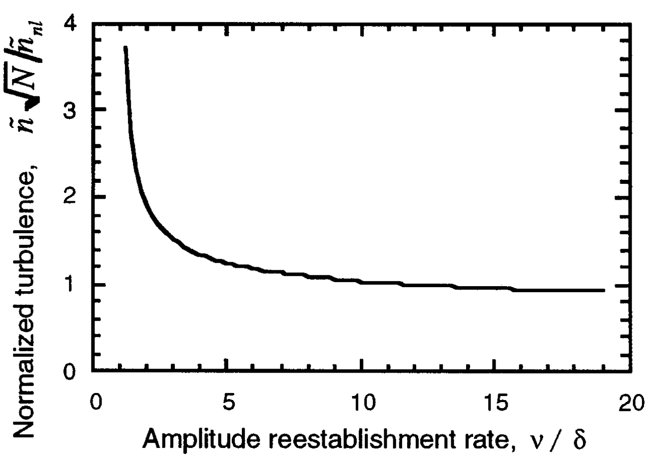

FIG. 2. The level of broadband turbulence required to make coincidental nonlinear events a major energy sink depends, as shown, upon the rate at which the mode amplitudes are reestablished following such an event. Nonlinear absorption dominates over linear damping in the region above the curve.

As one example, one can consider the introduction of one or more strongly driven modes into a system with a large, established level of broadband turbulence. This will tend, through mode coupling, to try to drive the broadband turbulence to larger amplitude. In addition, $\tilde{n}_{\rm nl}$ for the broadband turbulence will decrease because of the significant amplitude of the strongly driven modes. The resulting advent or increase of CNEs may increase the damping of the driven modes and may also drive down the level of broadband turbulence from its previous steady-state value. This may explain the observed reductions in the noise level [13,17] in laser plasmas when acoustic modes are driven by stimulated Brillouin scattering.

In conclusion, we have used a simple analysis to address the question of whether and when the interference of many, moderately damped modes may be important. We find that, once broadband turbulence reaches a level approximately equal to the nonlinear saturation amplitude divided by the square root of the number of interfering modes, there will be a significant probability of nonlinear saturation events. When the interfering modes are driven strongly enough, such events will significantly increase their damping. Further, these nonlinear events can influence the structure of the turbulence when one or more of the waves is strongly driven. These findings imply that a more detailed analysis of such effects is warranted in the many specific cases where such modes exist.

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[1] L. Rayleigh, *Theory of Sound* (Dover, New York, 1877).
[2] T. K. Caughey, J. Appl. Mech. **27**, 269 (1960).
[3] G. B. Warburton and S. R. Soni, Earthq. Eng. Struct. Dyn. **5**, 365 (1977).
[4] S. Y. Chen, M. S. Ju, and Y. G. Tsuei, Mech. Syst. Signal Process. **10**, 93 (1996).
[5] A. Craggs, J. Sound Vib. **170**, 130 (1994).
[6] C. K. Liu and J. E. Sneckenberger, J. Sound Vib. **177**, 43 (1994).
[7] J. M. Ziman, *Electrons and Phonons* (Clarendon, Oxford, 1960).
[8] Computer simulations can now be large enough that a

simulation designed to observe the effects discussed here could do so, by using enough particles and appropriate diagnostics.
[9] A. V. Maximov *et al.*, Phys. Plasmas **3**, 1689 (1996).
[10] H. A. Baldis *et al.*, Phys. Fluids B **5**, 3319 (1993).
[11] K. S. Bradley, Ph.D. thesis, University of California, Davis, 1993.
[12] S. D. Baton *et al.*, Phys. Rev. E **49**, R3602 (1994).
[13] R. P. Drake *et al.*, Phys. Rev. Lett. **74**, 3157 (1995).
[14] R. P. Drake, Comments Plasma Phys. Control Fusion **17**, 99 (1996).
[15] R. P. Drake, R. G. Watt, and K. Estabrook, Phys. Rev. Lett. **77**, 79 (1996).
[16] R. P. Drake, K. Estabrook, and R. G. Watt, Phys. Plasmas **4**, 1825 (1997).
[17] H. A. Baldis *et al.*, Phys. Rev. Lett. **77**, 2957 (1996).